ECONOMIC DESIGN OF FRACTION DEFECTIVE CONTROL CHARTS TO MAINTAIN CURRENT CONTROL OF A PROCESS

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TO MY MOTHER AND FATHER

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SUMMARY

The purpose of this investigation was to develop an expected minimum cost quality control model for the fraction defective control chart when there are S out of control states and the process is subject to random transitions between states. The time between process shifts is assumed to follow the exponential distribution. This objective was accomplished by treating the transition of the process between states as a finite Markov chain. The steady state probability of the process being in each state was found from a transition matrix, and the total expected cost associated with the quality control procedure was calculated. The solution gave the minimum cost sample size, interval between successive samples, and control chart limits.

An optimization technique based on the Hooke and Jeeves pattern search was developed and programmed for the digital computer. Numerical examples with various model parameters and cost coefficients were investigated, and the optimal values of the test parameters and expected cost were tabulated. Sensitivity of the optimal test parameters to changes in the model cost coefficients and parameters was also investigated.

The results of this investigation indicate that an attribute sampling plan deserves serious consideration in a wide variety of practical applications. As anticipated, the total expected cost associated with a fraction defective quality control procedure is greater than a similar quality control procedure based on measurement sampling. However, if either the fixed cost per sample or the expected mean shift of a process

is relatively large, or if the cost per unit sampled is less for an attribute sampling plan, then the difference between the total expected cost of the two sampling plans is smaller.

CHAPTER I

PROBLEM DESCRIPTION

1.1 Introduction

The purpose of this chapter is to provide an overview of the applications that control charts have in quality control, with emphasis on the fraction defective control chart used in this investigation. A brief survey of several types of control charts used to stabilize the output of production processes will be presented. The problem to be investigated will also be defined, and its place in the quality control literature will be indicated.

1.2 Statistical Quality Control

The goal of any quality control procedure is to accurately determine and to efficiently monitor the output of production processes. One such statistical procedure in quality control is based upon the use of control charts. The type of control chart employed depends upon the sampling technique, the sample test statistic, and the corrective actions to be taken upon recording the sample statistic.

Statistical analysis performed with control charts has proven to be of great importance when applied to problems evolving from the control of complex production processes. In the form of a graph, a control chart represents the current operating condition of a production process. If the output of a production process is assumed to be a random variable, then a statistic, that is, a function of the observations from a random sample, can be plotted on a control chart and the status of the system can be determined. A production process can be interpreted as producing only an expected number of defectives (in statistical control), or it can be interpreted as producing an unexpected number of defectives (out of statistical control).

When a process is said to be in control, the variation of the sample statistic is due only to random or chance causes. The amount of variation in the sample statistic may be predicted, but it cannot be traced to particular causes. When the system is said to be out of control, variation in the sample statistic does not conform to a pattern that might reasonably be produced by chance causes. The magnitude of this variation from the nominal, or in control value, indicates the presence of one or more assignable causes. Tolerance limits on the value of the sample statistic must be incorporated into a set of rules which establish the action to be taken upon evaluation of the sample statistic to insure the most efficient long run stability of the process. The magnitude of the variation in the sample statistic, above which assignable causes should be located and corrected, is an important question investigated in this thesis.

Control charts are frequently classified by the type of sampling employed and the test statistic. A frequently used procedure consists of selecting a random sample of N units at some appropriate interval of time, and determining if the specific units sampled are defective or non-defective. This type of sampling procedure is frequently called attribute sampling; that is, a unit is classified as either defective or non-defective on the basis of comparison with a standard. The alternative is measurements or variables sampling, in which the quality characteristic is measured on a numerical scale. Attributes sampling usually is performed in conjunction with a fraction defective control chart, or p chart. Measurements sampling usually is performed in conjunction with the $\overline{X}$ chart. Attribute sampling procedures often result in economy and simplicity in the inspection process, but to achieve an equal power to detect a shift in the process, they usually require a larger sample size than a corresponding measurement sampling plan.

The method used to classify items as either defective or non-defective is unique to the manufacturer's specification for the products produced. It is possible for a fairly complex product to have many defects, yet it may or may not be considered defective. However, the attribute sampling procedure must clearly stipulate each item sampled as either defective or non-defective.

As the quality control procedure utilized in this investigation involves the fraction defective control chart, it will be the only control chart discussed in detail. More detailed information on control charts is available in Hines and Montgomery (14) and Duncan (6).

1.3 The Fraction Defective Control Chart

Whenever it is possible to classify an item produced as either defective or non-defective on the basis of comparison with a standard, it may be desirable to utilize the fraction defective control chart as part of the quality control system. Each sample of size N may contain from 0 to N defective units, depending upon the definition of a defective unit and the status of the process. If we denote the number of defective units within a sample of size N by D, and assume D is a binomial random variable with known parameter N and unknown parameter p, then the sample fraction defective can be estimated by

$$\hat{p} = \frac{D}{N} ,$$

where

$$D \sim BIN(p,N) .$$

Furthermore, the variance of $\hat{p}$, $\sigma^2_{\hat{p}}$ is

$$\sigma^2_{\hat{p}} = \frac{\hat{p}(1 - \hat{p})}{N}$$

Let the center line of the p chart, denoted by p_0, represent the fraction defective of the process due to random error, that is, when the process is in control. Now the variance of $\hat{p}$ given that the true fraction defective $p = p_0$, is

$$\sigma^2_{p_0} = \frac{p_0(1 - p_0)}{N} .$$

Therefore, to construct the upper and lower control limits use

$$UCL = p_0 + L\sqrt{\frac{p_0(1 - p_0)}{N}} \text{ , and}$$

$$LCL = p_0 + L\sqrt{\frac{p_0(1 - p_0)}{N}} ,$$

where L represents a constant multiplier of the standard deviation of p_0. A typical p chart is shown in Figure 1.

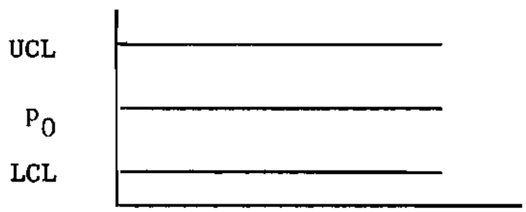


Figure 1. The p Chart

By definition, the fraction defective, p, for any process must be greater than or equal to zero. However, it is possible for the LCL to be less than zero. Should this occur, the LCL is assumed to equal zero.

The quality control procedure is essentially a test of the null hypothesis

$$H_0: \ p = p_0 ,$$

against the alternative hypothesis

$$H_1: p \neq p_0 .$$

The range space of all possible values of p is

$$0.0 \le p \le 1.0 \ .$$

The test statistic, $\hat{p}$, is a discrete random variable with range space $\{0, 1/N, 2/N, . . ., (N - 1)/N, 1\}$. Through the control limits, the range space of $\hat{p}$ is divided into two subsets. One subset containing those values of $\hat{p}$ which indicate the null hypothesis cannot be rejected, and the other subset containing those values of $\hat{p}$ which indicate the null hypothesis should be rejected. To accomplish this, we establish the following rule: the null hypothesis is not rejected unless the sample fraction defective falls above the upper control limit or below the lower control limit.

When the sample fraction defective falls within the control limits, we will assume the variation of $\hat{p}$ from p_0 can be explained by chance causes, and the process will be allowed to continue to operate. This indicates that there is no reason not to believe that the process is operating in the in control state defined by p_0. If the sample fraction defective falls into the critical region and the null hypothesis is rejected, then the variation of $\hat{p}$ from p_0 can no longer be explained by chance causes, and the process will be stopped and investigated for assignable errors. This indicates that the process is operating in one of the out of control states defined by p_i, $i = 1, 2, . . ., S$.

To define the fraction defective vector $\underline{p} = (p_0, p_1, p_2, . . ., p_S)$ for a process, we assume that the production process operates in a finite number of states $(S + 1)$ each defined by a unique fraction defective. We also assume the existence of only one in control state defined by p_0. With any production process, at least one assignable error will eventually cause a deterioration in the fraction defective from p_0 to some other value p_i. Should more than one assignable error cause similar variations in p_0, the probability of at least one occurring will define the probability of the process operating in a single out of control state. Therefore, the total number of out of control states is determined by the number of unique fractions defective identified. Many possible methods are available to model the transition of the process between states. The method used in this study will be discussed in Chapter II.

1.4 Survey of the Literature

Early applications of statistical quality control focused on methods employing a sample size chosen purely from statistical considerations. For example, a common procedure was to select the sample size so as to detect a given shift in the process with a prescribed power. The basic control chart designs, established by Shewart (20), dealt with sample sizes of four or five, control limits fixed at ± 3- sigma, and the interval between samples to be determined by the practitioner. As these classical concepts, such as minimizing both type I and type II errors, were important to early researchers the usage of small sample sizes and ± 3- sigma control limits became a traditional practice in statistical quality control.

The usual $\bar{X}$ chart is based on a normally distributed quality characteristic. The $\bar{X}$ chart with ± 3- sigma control limits has a probability of a type I error of approximately 0.0027. The p chart, based on the binomial distribution, with ± 3- sigma control limits has a type I error which depends upon the fraction defective and sample size. The type I error for the p chart can be several times larger than the type I error for the $\bar{X}$ chart. The $\bar{X}$ chart also has the advantage of providing a more powerful test (relative to the p chart) to detect a given shift in the process mean. As a result of these advantages, the $\bar{X}$ chart has become the most widely accepted technique used to control the long term stability of a production process.

These classical statistical concepts were long believed to be the basis of the design of a quality control procedure. More recently, researchers have attempted to consider both types of errors in terms of economics. Duncan (5), Cowden (4), and Girshick (11), defined the objective of a quality control procedure in economic terms.

Duncan (5) proposed a procedure for the univariate case to determine the sample size, interval between samples, and control limits, for an $\bar{X}$ chart which maximizes the average net income when a single assignable error exists. The form of Duncan's model is

$$\text{Profit} = \text{Income} - \text{Cost} .$$

If income is assumed to be independent of the quality control procedure and is considered a constant, then maximizing profit is equivalent to minimizing cost. Total cost is assumed to equal the sum of the average cost per hour of operating the quality control procedure, the average cost per hour of producing defectives, and the average cost per hour of a non-operative process. Duncan assumes that when a shift in the process occurs, it shifts by a constant amount; that the average time required for a shift to occur is $\frac{1}{\lambda}$; and that starting in a state of control at time = 0, the probability the process will still be in control at time t is $e^{-\lambda t}$. Goel et al. (12) have developed an algorithm for computing the optimal test parameters for Duncan's model.

Cowden (4) has developed a model for the economic design of a test procedure for controlling the mean of a process. The model minimizes a cost function which includes the cost of the test procedure, the cost of investigating the process, and the cost of producing defective items. Cowden assumes that the process is considered out of control at the start of each day. Once an assignable error is detected, it is immediately corrected, and no further errors can occur during that day. The cost of looking for an assignable cause is assumed to be proportional to the shift in the process mean. Finally, the probability of locating an assignable cause is assumed to be a function of the cost of looking for it.

More recently, Duncan (7) and Knappenberger and Grandage (16) have investigated the economic optimization of the $\bar{X}$ chart when several assignable errors exist. Duncan extended his earlier work to account for the occurrence of several assignable errors with known probability distributions. He indicated that the increased accuracy with multiple assignable errors is often negated by inaccurate estimates of the model's cost coefficients and parameters. As the effect of increasing the model's complexity may not provide more accurate results, he concluded that a single assignable cause model will suffice in many practical applications.

Knappenberger and Grandage (16) developed a comprehensive model to determine the long term expected cost associated with a quality control procedure. Perhaps the most comprehensive model based on a measurement sampling plan developed to date, it was chosen for use in this study. The detailed development of this model will be discussed in Chapter II and the importance of the underlying assumptions will be analyzed. While these assumptions require care in the proper use of this model, it is believed they are less restrictive than other available models.

In a recent study Baker (1) compared two alternative process models in the economic design of $\overline{X}$ charts. His research indicates that the assumption of an exponential process deterioration (Markov property) is tempting to use because of its simplicity, but it may lead to poor results in certain cases.

Taylor's (21,22) analysis of a univariate quality control model with one out of control state indicated that sampling should be determined at each stage by the current posterior probabilities, and that fixed sample sizes and sampling intervals yielded non-optimal results. While the non-optimal nature of this type of sampling plan is realized, such methods are widely used because of the ease by which they are adapted to practical situations. Consequently, the desirability of a uniform sampling technique will result in reduced administrative costs. The assumption to limit the sampling method to the class of fixed-sampling intervals made by Knappenberger and Grandage (16) and used in this investigation is desirable from a practical point of view and to maintain model simplicity.

The sampling procedure developed will be limited to quality control tests involving a single process parameter. Montgomery and Klatt (18) adopted the Knappenberger and Grandage model to the economic design of T^2 control charts, which is a multivariate analog of the $\overline{X}$ control chart.

They assumed the existence of only one assignable error and the time between random shifts in the process mean to be exponentially distributed.

The vast majority of research in this area has been devoted to $\overline{X}$ charts, with either one or several assignable causes. A limited amount of research has been directed toward multiple quality characteristics. A recent adaptation of Duncan's single assignable cause $\overline{X}$ chart model to the p chart has been presented by Ladany (17). This study develops the theory underlying the model, but no numerical results are reported.

1.5 Purpose and Scope

The theory underlying the use of control charts in quality control has been well developed. Many of the applications in univariate statistical quality control have involved the use of the $\overline{X}$ chart. While the $\overline{X}$ chart has the disadvantage of a more expensive sampling procedure, it does provide a more powerful test to detect a given shift in the process than does an attribute sampling procedure and a fraction defective control chart. The $\overline{X}$ chart is based on a normally distributed sample statistic and has been widely accepted and used in industry. The fraction defective control chart is based on a binomially distributed test statistic. In most industrial applications either the Poisson or normal approximation to the binomial is utilized.

The purpose of this investigation is to develop an expected minimum cost quality control model for the fraction defective control chart when there are S out of control states and the process is subject to random transitions between states. The solution will yield the optimal sample size, interval between samples, and control chart limits. Numerical examples of optimum parameter determination will be presented. Sensitivity of the optimal test parameters to changes in the model cost coefficients and parameters will also be investigated.

CHAPTER II

DEVELOPMENT OF THE MATHEMATICAL MODEL

2.1 General Assumptions and Nomenclature

This research will present a procedure to minimize the long term expected cost per unit associated with the control chart for the process fraction defective. It is assumed that the model can be adapted to a wide variety of production processes. In each case, the optimal design of the quality control procedure involves the determination of the following parameters: sample size (N), control chart limits (L), and interval between successive samples (K). While the above parameters will establish an optimal quality control procedure, their values will be dependent upon accurate estimation of various cost coefficients and other model parameters. These factors will be discussed in greater detail in later sections of this chapter.

The properties of a classical hypothesis testing procedure can be summarized by the probabilities of committing the two types of errors. A type I error occurs if the null hypothesis is rejected when it is true. In a quality control content, the cost of making this error involves the unnecessary investigation for assignable causes in the production process and the lost production during the investigation. A type II error occurs if the null hypothesis is not rejected when it is false. The costs of making this error involve the production of a larger percentage of defective units. Not only does this include the cost to the manufacturer of returning and replacing defective units, but also the cost incurred by the loss of future business due to erratic product quality. Usually, it is more convenient to characterize a test by its type I error and its power where the power of a test equals the probability of rejecting the null hypothesis when it is false.

For any mathematical model to represent the total cost associated with a quality control process, it is necessary to optimize the three test parameters in a manner that will minimize the sum of the costs incurred with committing either a type I or type II error and the costs associated with sampling. The economic feasibility of varying the test parameters is determined by the net change in the total expected cost per unit. The probability of committing both type I and type II errors can be reduced at the expense of increasing the sample size and decreasing the interval between samples. To decrease the cost of unnecessary investigation of the process, the probability of a type II error occurring can be reduced by increasing the size of the critical region (decrease L), but at the expense of increasing the probability of committing a type I error.

2.2 General Form of the Model

The mathematical model used to represent the total cost per unit related to a quality control procedure will be assumed to consist of the sum of three expected costs, and can be written as

$$E(C) = E(C_1) + E(C_2) + E(C_3) \tag{2.1}$$

where $E(C_1)$ is the expected cost per unit associated with the sampling and testing plan, $E(C_2)$ is the expected cost per unit of investigating and correcting the process when the null hypothesis is rejected, and $E(C_3)$ is the expected cost per unit associated with the production of defective products.

2.2.1 Expected Cost of Sampling and Testing

Duncan (5) and Knappenberger and Grandage (16) assume the cost of sampling and testing can be approximated by the sum of two cost components. The first cost, independent of the number of units sampled, is a constant amount per sample, while the second is the cost per unit sampled. Both cost factors are divided by the number of units produced between successive samples to obtain an average cost per unit of sampling and testing. As a constant is its own expected value, the expected cost of sampling and testing is assumed to be accurately approximated by

$$E(C_1) = \frac{a_1 + a_2 N}{K}, \tag{2.2}$$

where a_1 is the fixed cost per sample, a_2 is the cost per unit sampled, N is the number of units sampled, and K is the number of units produced between successive samples.

2.2.2 Expected Cost of Investigating and Correcting the Process (Rejecting H_0)

It has been widely accepted that the cost of investigating and correcting the cause of an apparent shift in the fraction defective from $p = p_0$ to some new value will depend upon the true value of p. However, the prior information needed to base the cost of investigating and correcting a process as a function of the true parameter p will not be available or will be most difficult to obtain.

Knappenberger and Grandage (16) assume that the cost of correcting the cause of small shifts is less than the cost of correcting the cause of larger shifts, and that the cause of small shifts requires more time to locate. Thus the cost of investigating small shifts is more expensive than with larger shifts. As the above costs tend to counteract each other, Knappenberger and Grandage (16) base the expected cost of investigating and correcting a process upon the generally available prior information concerning the number of times the process goes out of control, the length of time the process is stopped for repair, and the cost per hour (including repair costs) of an inoperative process. From this information the expected cost of investigating and correcting a process can be approximated adequately.

To evaluate this expected cost from the available prior information, several assumptions must be made. Assume there exists a random variable, V, with mean a_3, which approximates the expected cost of investigating and correcting a process, and with a distribution not dependent upon the true parameter p. If U is defined as another random variable which has the value one (1) when the null hypothesis is rejected and zero (0) otherwise, then we can express the expected cost per unit of rejecting H_0 as

$$E(C_2) = \frac{E(V)\cdot P(U = 1)}{K} = \frac{a_3 P(U = 1)}{K} .$$

For the above expression to be valid, it must be assumed that the cost of investigating and correcting a process is not incurred unless the null hypothesis is rejected and that the random variables are stochastically independent.

Define $\underline{p}$ as the row vector of probabilities, p_i, where p_i is the fraction defective probability for state i, i = 0, 1, 2,. . .,S. Let $\underline{q}$ be the row vector of probabilities, q_i, where q_i is the conditional probability of rejecting H_0 given the process is in state i ($p = p_i$) at the time the test is performed. Define $\underline{\alpha}$ as the row vector of probabilities, α_i, where α_i is the probability of the process being in state i ($p = p_i$) at the time the test is performed. Then the probability of rejecting H_0, $P(U = 1)$, equals the sum of the products of the corresponding components of vectors $\underline{q}$ and $\underline{\alpha}$, that is

$$P(U = 1) = \sum_{i=0}^{S} q_i \alpha_i = \underline{q}\,\underline{\alpha}^t ,$$

where the superscript t represents the transpose of the vector. Therefore, the expected cost per unit of rejecting the null hypothesis is

$$E(C_2) = \frac{a_3}{K}\,\underline{q}\,\underline{\alpha}^t . \tag{2.3}$$

2.2.3 Expected Cost of Producing Defectives (Accepting H_0)

The cost of producing defectives depends upon the criteria used to accept the null hypothesis. When H_0 is correctly accepted, a small percentage of defectives will be produced at some cost to the manufacturer. However, should H_0 be incorrectly accepted, the larger percentage of defectives produced will go undetected by the manufacturer to the buyer. If the buyer detects this larger percentage of defectives, then he may react several ways. He may reject just the defectives or reject the entire lot. Unwilling to accept products of poor quality, the buyer may seek another supplier. Other options are available to the buyer, but the one taken will affect the cost to the manufacturer. Because the exact relationship between the number of defectives and the cost to the manufacturer for each defective unit produced is difficult to determine and would result in unnecessary complications to the model, a simple linear relationship is assumed.

This is accomplished by defining a_4 as the cost to the manufacturer for each defective unit produced. Its value can be chosen to approximate the actual cost of producing a defective regardless of which state the process is operating.

Let W be defined as a random variable which has the value one (1) if a unit is defective and zero (0) if a unit is non-defective. Then the expected cost per unit of producing defects can be expressed as

$$E(C_3) = a_4 P(W = 1) \ .$$

The probability of producing a defective, P(W = 1), equals the sum of the products of the conditional probability of producing a defective unit given $p = p_i$ (fraction defective for state i) and the probability that the process is in state i where i = 0, 1, 2,. . .,S.

If we define, $\underline{\gamma}$ as the row vector of probabilities γ_i, where γ_i is the probability that the process is in state i, then

$$P(W = 1) = \sum_{i=0}^{S} p_i \gamma_i = \underline{p}\ \underline{\gamma}^t \ .$$

Therefore, the expected cost per unit associated with the production of defective units is

$$E(C_3) = a_4 \underline{p}\, \underline{\gamma}^t . \qquad (2.4)$$

2.2.4 Expected Cost Model

Combining equations (2.1), (2.2), and (2.3) and (2.4), the model becomes

$$E(C) = \frac{a_1 + a_2 N}{K} + \frac{a_3}{K} \underline{q}\, \underline{\alpha}^t + a_4 \underline{p}\, \underline{\gamma}^t . \qquad (2.5)$$

To optimize the above equation, it is necessary to find the optimal values for the test parameters N, L, and K. The a_i's, i = 1, 2, 3, and 4, were defined to be independent of the test parameters. The vector $\underline{p}$ depends on the fractions defective for the different states and the definition of a defective unit, but it is not dependent upon the test parameters. The vectors $\underline{q}$, $\underline{\alpha}$, and $\underline{\gamma}$ depend upon values of the test parameters. A complete explanation of this functional dependency upon the test parameters will be discussed in later sections.

2.3 Development of the Probability Vectors

The purpose of this section is to complete the development of the vectors $\underline{q}$, $\underline{\alpha}$, and $\underline{\gamma}$.

2.3.1 Development of the Vector $\underline{q}$

As defined earlier, $\underline{q}$ is the row vector of probabilities q_i, where q_i is the probability of rejecting H_0 given $p = p_i$, $i = 0, 1, \ldots, S$, at the time the test is performed, or q_0 is the probability of a type I error, and q_i is the power of the test to detect the shift of p from $p = p_0$ to some other value $p = p_i$, $i = 1, 2, \ldots, S$.

The critical region, defined in terms of the test parameter L and the standard deviation of p_0, depends on the probability distribution of the sample statistic. Since this study is limited to attribute sampling (p chart), it will be assumed that the sample statistic, $\hat{p} = \frac{D}{N}$, is obtained from the number of defectives (D) in a sample of size N. If D is assumed to follow a binomial distribution with parameters p_i (fraction defective for state i, $i = 0, 1, \ldots, S$) and the sample size is N, then the size of the critical region can be defined such that the null hypothesis is rejected if

$$\hat{p} \geq UCL = p_0 + L\sqrt{p_0(1 - p_0)/N} \text{ or } \hat{p} \leq LCL = p_0 - L\sqrt{p_0(1 - p_0)/N} .$$

Using the definition of the sample statistic, $\hat{p}$, we see that the null hypothesis is rejected if

$$N\hat{p} \geq N(UCL) + L\sqrt{p_0(1 - p_0)N} \text{ or } N\hat{p} \leq N(LCL) - L\sqrt{p_0(1 - p_0)N} .$$

Thus, the probability of rejecting H_0 when the process is in state 0 (in control) at the time the test is performed is

$$q_0 = P(\text{Reject } H_0 \mid p = p_0)$$

$$q_0 = P(\hat{p} \geq UCL \mid p = p_0) + P(\hat{p} \leq LCL \mid p = p_0)$$

$$q_0 = P(D \geq N(UCL) \mid p = p_0) + P(D \leq N(LCL) \mid p = p_0) \; , \tag{2.6}$$

and the probability of rejecting H_0 when the process is in state i, i = 1, 2,. . .,S, (out of control) at the time the test is performed is

$$q_i = P(\text{Reject } H_0 \mid p = p_i)$$

$$q_i = P(\hat{p} \geq UCL \mid p = p_i) + P(\hat{p} \leq LCL \mid p = p_i)$$

$$q_i = P(D \geq N(UCL) \mid p = p_i) + P(D \leq N(LCL) \mid p = p_i) \; . \tag{2.7}$$

When the control limits, N(UCL) and N(LCL), are integers, the evaluation of equations (2.6) and (2.7) for the components of the vector $\underline{q}$ is straightforward. As it is reasonable to assume one or both control limits will not be integers, some additional assumptions must be made to clarify the size of the critical region.

As D, by definition, must equal an integer value between 0 and N inclusive, the control limits must be rounded to equal an allowable value of D to properly define the size of the critical region. The following three rules outline the procedure to deal with those instances when the control limits do not have integer values.

(1) If the upper control limit, N(UCL), does not equal an integer, then its value must be rounded up to the next largest integer. This will be denoted as $[N(UCL)]^+$. Now, we assume the portion of the critical region defined by $[N(UCL)]^+$ equals the probability of the event $D = ([N(UCL)]^+, \ldots, N)$ and the null hypothesis is rejected if $[N(UCL)]^+ \leq D \leq N$ occurs.

(2) If the lower control limit, $N(LCL) < 0$, then by an earlier assumption it will be rounded up to equal zero. This will be denoted as $[N(LCL)]^+$. When this occurs, we assume the portion of the critical region defined by $[N(LCL)]^+$ equals zero (0). Usually the null hypothesis is rejected when $D \le N(LCL)$, but this is impossible when $N(LCL) < 0$; therefore, $P(D = 0)$ is not included in the critical region, because it would artificially increase both the type I error and the power of the test.

(3) If the lower control limit, $N(LCL) > 0$ and does not equal an integer, then its value must be rounded down to the next smallest integer. This will be denoted as $[N(LCL)]^-$. Now we assume the portion of the critical region defined by $[N(LCL)]^-$ equals the probability of the event $D = (0, 1, . . ., [N(LCL)]^-)$ and the null hypothesis is rejected if $0 \le D \le [N(LCL)]^-$ occurs.

Therefore, the components of the vector $\underline{q}$ equal

$$q_0 = \sum_{D=[N(UCL)]^+}^{D=N} \binom{N}{D}(p_0)^D(1 - p_0)^{N-D} + \sum_{D=0}^{D=[N(LCL)]^-} \binom{N}{D}(p_0)^D(1 - p_0)^{N-D} \text{ , and} \tag{2.8}$$

$$q_i = \sum_{D=[N(UCL)]^+}^{D=N} \binom{N}{D}(p_i)^D(1 - p_i)^{N-D} + \sum_{D=0}^{D=[N(LCL)]^-} \binom{N}{D}(p_i)^D(1 - p_i)^{N-D} . \tag{2.9}$$

$i = 1, 2, . . ., S$

It is understood that the second term in both expressions equals zero when $N(LCL) < 0$.

2.3.2 Development of the Vector $\underline{\alpha}$

As defined earlier, $\underline{\alpha}$ is the row vector of probabilities, α_i, where

α_i is the probability of the process being in state i ($p = p_i$) at the time the test is performed. To determine $\underline{\alpha}$, a transition probability matrix, B, is required. The elements of B, b_{ij}, represent the probability the process shifts from state i to state j during the production of K units between successive samples. To determine the probability of the process being in the different states after the production of K units, the a priori probability vector $\underline{c}$ must be defined. Let $\underline{c}$ be a row vector of probabilities, c_i, where c_i is the probability the process will shift from the in control state ($p = p_0$) to the out of control state ($p = p_i$) during the production of K units.

If we assume the time a process remains in control before going out of control is an exponential random variable with mean λ^{-1} hours, then the probability of remaining in control for h hours is

$$c_0 = 1 - \int_0^h e^{-\lambda t} dt = e^{-\lambda h} .$$

With the production of fractional units allowed and with R units produced per hour, we can assume the system will take h hours to produce K units as

$$h = K/R .$$

Replacing h with K/R, c_0 becomes

$$c_0 = e^{-\lambda K/R} .$$

Since λ^{-1} has been defined as the mean time, in hours, before a shift occurs and R the production rate of the process in units/hour, the following substitution, $\lambda/R = \lambda'$, expresses c_0 in terms of the average number of units produced before a shift occurs. Then

$$c_0 = e^{-\lambda' K} , \tag{2.10}$$

where λ' is the average number of units produced before a shift from the in control state occurs.

As $\underline{c}$ must satisfy the constraint

$$\sum_{i=0}^{S} c_i = 1 ,$$

Knappenberger and Grandage (16) choose to distribute the remaining probability, $1 - e^{-\lambda' K}$, over the S out of control states in the following way.

The general form of the binomial probability of i successes in S trials is

$$c_i = \frac{S!}{i!(S - i)!} \pi^i (1 - \pi)^{S-i} , \tag{2.11}$$

where $\pi = P(\text{Success})$ and $0.0 < \pi < 1.0$. By definition, $\underline{c}$ is subject to the constraint $c_i \geq 0$, for all i's. Therefore,

$$\sum_{i=1}^{S} c_i = 1 - c_0 = 1 - (1 - \pi)^S ,$$

by substituting $i = 0$ into equation (2.11). As the remaining probability to be distributed over the S out of control states equals

$$\sum_{i=1}^{S} c_i = 1 - e^{-\lambda' K} ,$$

the out of control probabilities, c_i, must be scaled in the following way:

$$c_i = \frac{(1 - e^{-\lambda' K}) S! \pi^i (1 - \pi)^{S-i}}{(1 - (1 - \pi)^S) i! (S - i)!}, \quad i = 1, 2, . . ., S . \qquad (2.12)$$

Notice that the constraints are satisfied by the above values of c_i.

In addition to the simplicity of this method, the components of vector $\underline{c}$ can be regulated by different choices of λ', S, and π to approximate the *a priori* distribution of out of control states for a wide variety of production processes.

With both vectors $\underline{q}$ and $\underline{c}$ defined, the transition matrix B can be structured. The elements of B, say b_{ij}, are the probabilities that the system will go from state i to state j during the production of K units. The definition of a transition matrix requires that:

(1) $\sum_{j=0}^{S} b_{ij} = 1$, for all i's

and

(2) $0 < b_{ij} < 1$, for all i, j's.

As the process has one in control state and S out of control states, the transition matrix B will be a square matrix of order S + 1. To simplify the calculation of the elements, b_{ij}, three assumptions are made to define allowable transitions:

(1) Once a process goes out of control, it stays out of control until detected or until H_0 is rejected.

(2) Once a process goes out of control, it will not correct itself and will be corrected only when H_0 is rejected. If not corrected, the process may shift to a higher out of control state.

(3) During the production of K units between successive samples, only one shift is permitted during the K/R hours.

The transition matrix B with S out of control states is of the form

$$B = \begin{array}{cc} & \text{(state j at time = t+K/R hrs.)} \\ \begin{array}{c}\text{(state i at time =} \\ \text{t hrs.)}\end{array} & \begin{bmatrix} b_{00} & b_{01} \cdots & b_{0j} & \cdots b_{0S} \\ & & \vdots & \\ b_{i0} & \cdots & b_{ij} & \cdots b_{iS} \\ & & \vdots & \\ b_{S0} & b_{S1} \cdots & b_{Sj} & \cdots b_{SS} \end{bmatrix} \end{array}$$

To calculate the individual terms of B, b_{ij}, four different cases must be investigated:

(1) When $0 \le j < i$, $i = 1, 2, \ldots, S$, b_{ij} equals:

b_{ij} = P(Process is in state i at time t) $\cdot$ P(Process is in state j at time t+K/R) = P(Reject H_0 at time t $|$ $p = p_i$) $\cdot$ P(Process shifts from state 0 to state j during the production of K units),

$$b_{ij} = q_i c_j \ . \tag{2.13}$$

(2) When $i = 0$ and $j = 0, 1, \ldots, S$, b_{ij} equals:

b_{ij} = P(Process shifts from state 0 to state j during the production of K units),

$$b_{ij} = c_j \ . \tag{2.14}$$

(3) When $i = j$, $i = 1, 2, \ldots, S$, b_{ij} equals:

$b_{ij} = b_{ii}$ = P(Reject H_0 at time t $|$ $p = p_i$) $\cdot$ P(Process returns from state 0 to state i during the production of K units) + P(Fail to reject H_0 at time t | $p = p_i$) · P(Process remaining in state i during the production of the next K units) ,

$$b_{ij} = b_{ii} = q_i c_i + (1 - q_i)P_{ii}, \text{ where } P_{ii} = \sum_{j=1}^{i} c_j/(1 - c_0) \text{ , or}$$

$$b_{ij} = b_{ii} = q_i c_i + (1 - q_i) \sum_{j=1}^{i} c_j/(1 - c_0) \ . \tag{2.15}$$

(4) When $0 \leq i < j$, $j = 3, 4, . . ., S$, b_{ij} equals:

b_{ij} = P(Reject H_0 at time t | $p = p_i$) · P(Process shifts from state 0 to state j during the production of K units) + P(Fail to reject H_0 at time t | $p = p_i$) · P(Process shifts, without returning to state 0, directly from state i to state j during the production of K units),

$$b_{ij} = q_i c_j + (1 - q_i)P_{ij}, \text{ where } P_{ij} = c_j/(1 - c_0), \text{ or}$$

$$b_{ij} = q_i c_j + (1 - q_i)c_j/(1 - c_0) \ . \tag{2.16}$$

Now B is the transition matrix of an irreducible, aperiodic, positive recurrent Markov chain. Therefore, $\underline{\alpha}$, the long run (steady state) unconditional probability of being in state j, $j = 0, 1, . . ., S$, can be calculated from the equation $\underline{\alpha}B = \underline{\alpha}$. The B matrix for S = 6 out of control states is shown in Figure 2.

Because $\underline{\alpha}$ is a probability vector, it must satisfy the constraint

$$\sum_{j=0}^{S} \alpha_j = 1, \text{ where } 0 < a_j < 1 \text{ for } j = 0, 1, . . ., S \ .$$

Rewriting the equation $\underline{\alpha}B = \underline{\alpha}$ will yield

$$\underline{\alpha}B - \underline{\alpha} = \underline{0} \text{ , or}$$

$$\underline{\alpha}(B - I) = \underline{0} \ , \tag{2.17}$$

where I is a (S + 1) identity matrix and $\underline{0}$ is a row vector containing (S + 1) zeroes.

The constraint $\sum_{j=0}^{S} \alpha_j = 1$ is satisfied by replacing the $(S + 1)^{st}$ column of the matrix (B - I) with a column of one's and replacing the $(S + 1)^{st}$ zero of vector $\underline{0}$ with a one. The modified form of equation (2.17) can be written as

$$\underline{\alpha}(B - I|1) = (\underline{0|1}) \ . \tag{2.18}$$

Right multiplying both sides of the above equation by the inverse of matrix $(B - I|1)$ yields

$$\underline{\alpha}(B - I|1)(B - I|1)^{-1} = (\underline{0|1})(B - I|1)^{-1} \ , \text{ or}$$

$$\underline{\alpha} = (\underline{0|1})(B - I|1)^{-1} \ . \tag{2.19}$$

Denote the elements of the inverse matrix $(B - I|1)^{-1}$ as $b^{*\,-1}_{ij}$. By carrying out the implied multiplication in equation (2.19), the vector $\underline{\alpha}$ equals the $(S + 1)^{st}$ row of the matrix $(B - I|1)^{-1}$. Thus, for $j = 0, 1, \ldots, S$,

$$\alpha_j = b^{*\,-1}_{S,j} \ .$$

2.3.3 Development of the Vector $\underline{\gamma}$

To accurately determine the cost of producing defectives, the vector $\underline{\alpha}$ must be modified to account for the process changing states at times other than when the test is performed. The vector $\underline{\alpha}$ represents the steady state probability of the process being in state i at the time the test is performed, however, this restriction on the vector $\underline{\alpha}$ must be

State j (time = t + K/R)

$$
B = \begin{array}{c|ccccccc}
 & \underline{0} & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} \\
\hline
\underline{0} & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
\underline{1} & q_1c_0 & q_1c_1+\frac{(1-q_1)c_1}{(1-c_0)} & q_1c_2+\frac{(1-q_1)c_2}{(1-c_0)} & q_1c_3+\frac{(1-q_1)c_3}{(1-c_0)} & q_1c_4+\frac{(1-q_1)c_4}{(1-c_0)} & q_1c_5+\frac{(1-q_1)c_5}{(1-c_0)} & q_1c_6+\frac{(1-q_1)c_6}{(1-c_0)} \\
\underline{2} & q_2c_0 & q_2c_1 & q_2c_2+\frac{(1-q_2)(c_1+c_2)}{(1-c_0)} & q_2c_3+\frac{(1-q_2)c_3}{(1-c_0)} & q_2c_4+\frac{(1-q_2)c_4}{(1-c_0)} & q_2c_5+\frac{(1-q_2)c_5}{(1-c_0)} & q_2c_6+\frac{(1-q_2)c_6}{(1-c_0)} \\
\underline{3} & q_3c_0 & q_3c_1 & q_3c_2 & q_3c_3+\frac{(1-q_3)(c_1+c_2+c_3)}{(1-c_0)} & q_3c_4+\frac{(1-q_3)c_4}{(1-c_0)} & q_3c_5+\frac{(1-q_3)c_5}{(1-c_0)} & q_3c_6+\frac{(1-q_3)c_6}{(1-c_0)} \\
\underline{4} & q_4c_0 & q_4c_1 & q_4c_2 & q_4c_3 & q_4c_4+\frac{(1-q_4)(c_1+c_2+c_3+c_4)}{(1-c_0)} & q_4c_5+\frac{(1-q_4)c_5}{(1-c_0)} & q_4c_6+\frac{(1-q_4)c_6}{(1-c_0)} \\
\underline{5} & q_5c_0 & q_5c_1 & q_5c_2 & q_5c_3 & q_5c_4 & q_5c_5+\frac{(1-q_5)(c_1+c_2+c_3+c_4+c_5)}{(1-c_0)} & q_5c_6+\frac{(1-q_5)c_6}{(1-c_0)} \\
\underline{6} & q_6c_0 & q_6c_1 & q_6c_2 & q_6c_3 & q_6c_4 & q_6c_5 & q_6c_6+\frac{(1-q_6)(c_1+c_2+c_3+c_4+c_5+c_6)}{(1-c_0)}
\end{array}
$$

State i (time = t)

Figure 2. The Transition Matrix B for S=6

removed. As defined earlier, $\underline{\gamma}$ is the row vector of probabilities γ_i, where γ_i is the probability that the process is in state i. To calculate the components of vector $\underline{\gamma}$, the probability of a shift from one state to another occurring between successive samples must be determined.

Given the time before a process shifts out of control is an exponential random variable with mean λ^{-1} hours, Duncan (7) showed that the average time elapsed (FR), during the h hour interval between the u^{th} and $(u+1)^{st}$ samples, before a shift occurs can be expressed as

$$FR = \frac{\int_{uh}^{(u+1)h} e^{-\lambda t}(t - uh)dt}{\int_{uh}^{(u+1)h} e^{-\lambda t}\lambda dt} = \frac{e^{-\lambda uh}\int_{0}^{h} e^{-\lambda t}\lambda t dt}{e^{-\lambda uh}\int_{0}^{h} e^{-\lambda t}\lambda dt}\text{, or}$$

$$FR = \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda(1 - e^{-\lambda h})}, \tag{2.20}$$

where h is the number of hours to produce K units, and λ^{-1} is the mean time, in hours, before a shift occurs. Dividing equation (2.20) by h, we obtain the average fraction of the period (F) between successive samples before a shift occurs, or

$$F = \frac{FR}{h} = \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda h(1 - e^{-\lambda h})}.$$

To express F as a function of the units produced between samples (K) and the production rate (R), let $\lambda' = \lambda/R$ and $h = K/R$. Thus,

$$F = \frac{1 - (1 + \lambda' K)e^{-\lambda' K}}{\lambda' K(1 - e^{-\lambda' K})} , \tag{2.21}$$

where K is the number of units produced between samples and λ' is the average number of units produced by the process before a shift from the in control state occurs.

To satisfy the earlier assumption that the process is unable to correct itself from an out of control state to the in control state, γ_0, the in control steady state probability is γ_0 = P(Process is in control when the u^{th} sample is taken) · P(Process remains in control until the $(u+1)^{st}$ sample is taken) + P(Process is in control when the u^{th} sample is taken) · P(Process shifts to an out of control state during the production of K units between the u^{th} and $(u+1)^{st}$ samples), or

$$\gamma_0 = \alpha_0 c_0 + F\alpha_0(1 - c_0) . \tag{2.22}$$

The steady state probabilities for state i = 1, 2,. . .,S, are γ_i = P(Process is in state i when the u^{th} sample is taken) · P(Process remains in state i until the $(u+1)^{st}$ sample is taken) + P(Process is in state 0, in control, when the u^{th} sample is taken) · P(Process shifts to an out of control state i during the production of K units between the u^{th} and $(u+1)^{st}$ samples) + P(Process is in some lower out of control state m, where $m < i$, when the u^{th} sample is taken) · P(Process shifts directly from state m to state i during the production of K units between the u^{th} and $(u+1)^{st}$ samples) + P(Process is in state i when the u^{th} sample is taken) · P(Process shifts to some higher out of control state r, where $r > i$, during the production of K units between the u^{th} and $(u+1)^{st}$

samples). This can be expressed as

$$\gamma_i = \frac{\alpha_i(\sum_{j=1}^{i} c_j)}{(1 - c_0)} + \alpha_0(1 - F)c_i + \sum_{m=1}^{i-1} \alpha_m(\frac{c_i}{1 - c_0})(1 - F) + \frac{\alpha_i F}{(1 - c_0)} \sum_{r=i+1}^{S} c_r \ . \quad (2.23)$$

While equation (2.23) is the general form of γ_i, $i = 1, 2, . . ., S$, two restrictions on its use are necessary to satisfy the constraints defining allowable transitions among the states.

(1) A transition to state 1 is possible only from the in control state (0). Therefore, the third term in equation (2.23) equals zero whenever $i = 1$.

(2) Once the system shifts to the highest out of control state (S), a shift to another out of control state is impossible. Therefore, the fourth term in equation (2.23) equals zero whenever $i = S$.

2.4 Optimization Technique

The method used to minimize the total expected cost associated with the quality control procedure is the Hooke and Jeeves pattern search. It is a sequential search routine for minimizing a function, say f, of a vector-valued variable $\underline{X}$. For our problem, $\underline{X} = (N, L, K)$ is a three-dimensional vector with the components of $\underline{X}$ equal to the test parameters. A discussion of the Hooke and Jeeves pattern search can be obtained from Fan, Erickson, Hwang (8).

Before the Hooke and Jeeves pattern search could be used to optimize equation (2.5), some modifications to this technique were necessary. These modifications restricted the allowable values of the test statistics in the vector $\underline{X} = (N, L, K)$ to a set of points defined as feasible by the model assumptions. These restrictions can be summarized as follows:

(1) Two of the parameters, N and K, were defined in the model as integers; therefore, a feasible value of $\underline{X}$ was restricted to integers values of N and K.

(2) When the distribution of the sample statistic is continuous, L and the control limits must be continuous. However, sample statistic $\hat{p} = D/N$ used with fraction defective control chart has a discrete range space. If we assume that the null hypothesis is rejected if $\hat{p}$ falls on or outside the control limits, then the control limits can be restricted to values in the range space of $\hat{p}$. The control chart limits are evaluated by equation (2.5) and rounded according to the rules in section 2.3.1. The size of the critical region in this model does not change continuously, but it does change by increments when the rounded control limits change. While this procedure of rounding the control limits to discrete values leaves the value of the control chart limits dependent upon L, they are no longer functions of L. This allows L to vary continuously within some limited range, L_{min} to L_{max}, without changing $[N(UCL)]^{+}$, $[N(LCL)]^{-}$ or the size of the critical region. The pattern search was modified to require the minimum allowable variation in L to be sufficiently large to insure a change in the size of the critical region. Thus, a change in L of this magnitude will affect the type I error and the power of the test.

(3) An allowable change in $\underline{X}$ was defined as the variation in the test statistics between feasible values.

(4) The initial value of $\underline{X}$ and the end points of the allowable range space of the test statistics were restricted to feasible values of

the test statistics.

A flow diagram for the Hooke and Jeeves Pattern search is shown in Figure 3.

The global optimum can be found with the pattern search if the function f is convex. Some analysis of the behavior of the cost surface has been conducted which indicates that the surface is approximately convex in a limited region around the optimal. The convexity of the surface cannot be proven. However, within a limited range of the test parameters, the surface was assumed to be convex. The behavior of the cost surface will be discussed in section 3.6.

2.5 Numerical Example

To illustrate the use of equation (2.5) to design an optimal sampling plan consider the following example taken from Table 5.

$a_1 = \$\ 5.0$

$a_2 = \$\ 0.1$

$a_3 = \$20.0$

$a_4 = \$10.0$

$\lambda' = 1000$

$S = 6$

$\underline{p} = (.01,\ .02,\ .04,\ .08,\ .16,\ .32,\ .64)$

$\pi = .376$

For this example, the optimal sampling plan is

$N = 14$

$L_{max} = 2.31$

$K = 81$

Since,

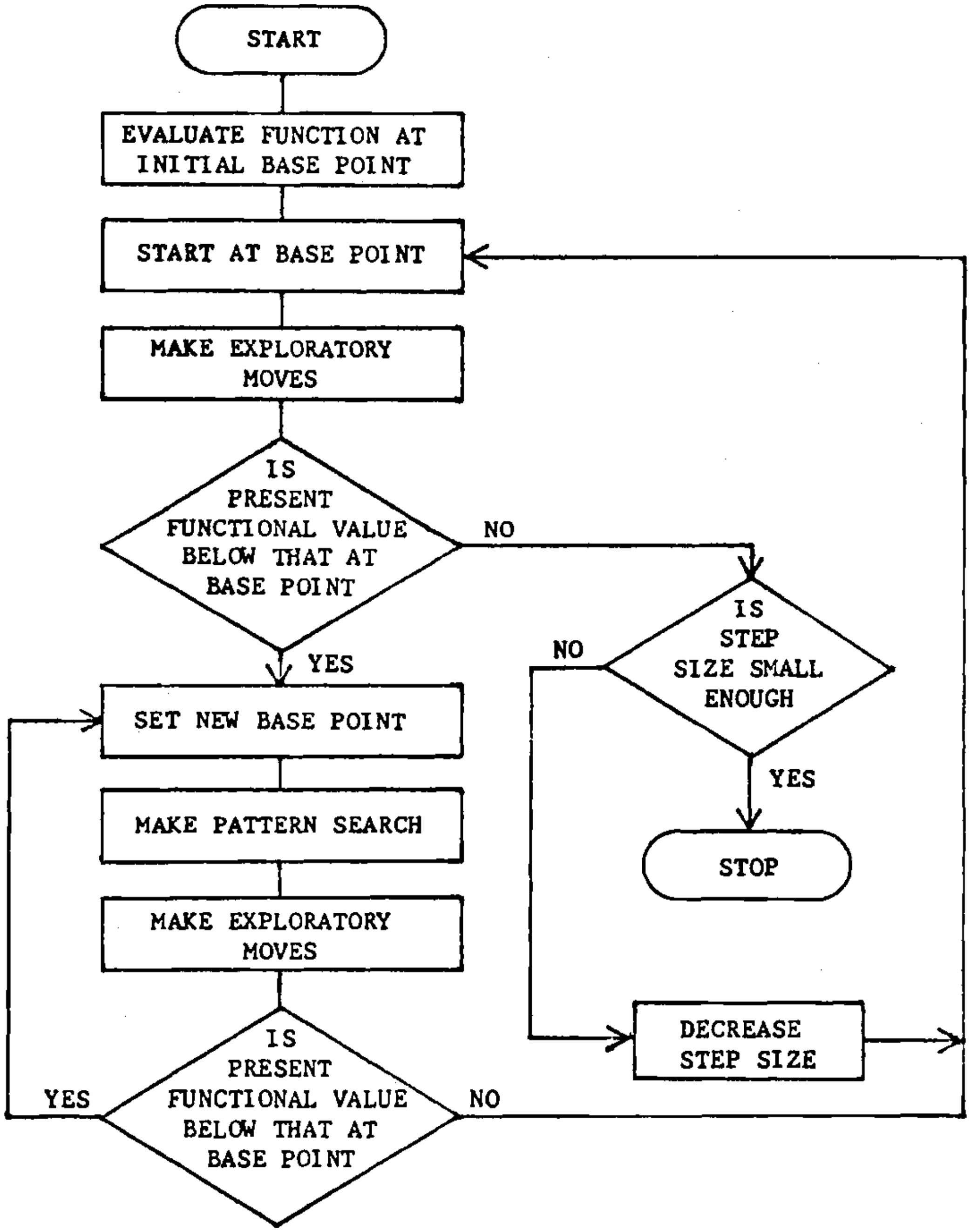


Figure 3. Flow Diagram For the Hooke and Jeeves Pattern Search

$$N(UCL) = Np_0 + L\sqrt{p_0(1 - p_0)N}$$

$$N(UCL) = 14(.01) + 2.31(.3723) = 1.0$$

$$UCL = N(UCL)/N = 0.071$$

and

$$N(LCL) = Np_0 - L\sqrt{p_0(1 - p_0)N}$$

$$N(LCL) = 14(.01) - 2.31(.3723) = -0.72$$

$$LCL = N(LCL)/N = -0.051 ,$$

the optimal sampling procedure is to take a sample of 14 units every 81 units produced and reject H_0 if

$$\hat{p} \geq UCL = 0.071 , \text{ or } \hat{p} \leq LCL = -0.051 .$$

The $P(\hat{p} < 0) = 0$. Therefore, reject H_0 if $\hat{p} \geq 0.071$, or if one defective unit is found in the sample of 14 units. The expected cost per unit associated with the optimal control procedure is

$$E(C) = \$0.3498$$

For this example with $N = 14$, $L_{max} = 2.31$, and $K = 81$, we find that $\underline{c} = (.9222, .0176, .0266, .0214, .0097, .0023, .0002)$, and $\underline{q} = (.1313, .2464, .4353, .6888, .9129, .9955, 1.000)$.

The modified transition matrix defined by equation (2.18) is

$$(B - I|1) = \begin{bmatrix} -.0778 & .0176 & .0266 & .0214 & .0097 & .0023 & 1.0 \\ .2272 & -.8247 & .2641 & .2122 & .0959 & .0231 & 1.0 \\ .4015 & .0077 & -.6674 & .1643 & .0743 & .0179 & 1.0 \\ .6352 & .0122 & .0183 & -.7229 & .0453 & .0109 & 1.0 \\ .8419 & .0161 & .0243 & .0195 & -.9070 & .0047 & 1.0 \\ .9180 & .0176 & .0265 & .0213 & .0096 & -.9932 & 1.0 \\ .9222 & .0176 & .0266 & .0214 & .0097 & .0023 & 1.0 \end{bmatrix}$$

The bottom row of the inverse of the above matrix equals the vector $\underline{\alpha}$. Thus from equation (2.19)

$$\underline{\alpha} = (.8714,\ .0201,\ .0446,\ .0424,\ .0172,\ .0039,\ .0004).$$

The vector $\underline{\gamma}$ defined by equations (2.22) and (2.23) equals

$$\underline{\gamma} = (.8371,\ .0200,\ .0501,\ .0574,\ .0279,\ .0068,\ .0007).$$

To compute the expected costs per unit for this example use equations (2.1), (2.2), (2.3), and (2.4).

$$E(C_1) = \frac{a_1 + a_2 N}{K} = \frac{5 + 0.1(14)}{81} = \$0.079$$

$$E(C_2) = \frac{a_3}{K}\,(\underline{q}\ \underline{\alpha}^t) = \frac{20}{81}\,(0.03248) = \$0.0464$$

$$E(C_3) = a_4(\underline{p}\ \underline{\gamma}^t) = 10(0.02244) = \$0.2244$$

$$E(C) = E(C_1) + E(C_2) + E(C_3) = \$0.3498$$

CHAPTER III

NUMERICAL RESULTS

3.1 A Comparison with the $\bar{X}$ Chart

The results of this section are for several values of the cost coefficients (a_1, a_2, a_3, and a_4), three values of the a priori distribution parameter (π), and S = 6 out of control states. To obtain these results, the fractions defective vector, $\underline{p}_N$, associated with the different states is defined so as to agree with the Knappenberger and Grandage (16) model of the $\bar{X}$ chart.

Let a defective unit be defined as a unit whose quality characteristic falls outside the interval $\mu_0 \pm 3\sigma$, where μ_0 is the in control process mean and σ is the process standard deviation. If the out of control states are defined as $\mu_i = \mu_0 \pm i\sigma$, where $i = 1, 2, \ldots, S$, then P(Producing a defective unit$|\mu = \mu_i$) equals the fraction defective for state i. These are shown in Table 1.

Table 1. Development of the Fraction Defective Vector $\underline{p}_N$

State	Mean	P(Producing a defective unit$\|\mu = \mu_i$) = p_i
0	μ_0	.0027
1	$\mu_0 + 1\sigma$	.0228
2	$\mu_0 + 2\sigma$	.1587
3	$\mu_0 + 3\sigma$	.5000
4	$\mu_0 + 4\sigma$	.8413
5	$\mu_0 + 5\sigma$	.9773
6	$\mu_0 + 6\sigma$	.9987

By using the vector $\underline{p} = \underline{p}_N$, a direct comparison can be made between the model developed in section 2.2 and the model of the $\bar{X}$ chart developed by Knappenberger and Grandage (16). The comparison, shown in Table 2, with parameters $a_4 = 10$, $\lambda' = 1000$, $S = 6$, and $\underline{p} = \underline{p}_N$, allows direct comparison of the minimum expected cost and the optimal values of the test parameters between the $\bar{X}$ chart and p chart models for different values of a_1, a_2, a_3, and π.

The most interesting result of this comparison is the similarity between the optimal solutions based on the two models. The expected cost per unit of an attribute sampling plan is generally more expensive. This might be expected, as the p chart produces a smaller power of the test to detect a given shift and allows more defectives to go undetected than the $\bar{X}$ chart. This difference in cost may also be explained by the larger samples generally used by the p chart.

The degree to which the $\bar{X}$ chart model has an economic advantage over the p chart model depends on the relative values of the cost coefficients and the model parameter π. This economic advantage of the $\bar{X}$ chart tends to decrease when either a_1 or π increases. When both a_1 and π increase, the economic advantage of the $\bar{X}$ chart is further decreased. However, the results shown in Table 2 indicate the economic advantage changes in favor of the p chart.

This change in economic efficiency when $a_1 = 1000$ and $\pi = .800$ was not expected to occur, as the $\bar{X}$ chart will always have an economic advantage to detect a given shift in the process with a prescribed power relative to the p chart. Therefore, this change in economic efficiency could indicate an error in the optimal solution of the $\bar{X}$ chart model due to the optimization technique employed.

Table 2. Comparison Between the $\bar{X}$ Chart and the p Chart with $\underline{p}=\underline{p}_N$, $a_4=10$, $\lambda'=1000$, and $S=6$

				$\pi=.376$			$\pi=.597$			$\pi=.800$		
				a_1			a_1			a_1		
a_2	a_3	Chart	Parameters	1.0	10	100	10	100	1000	10	100	1000
1.0	10	p	E(C)				.8007	2.1309		.8527		
			N				2	6		2		
			L_{max}				13.5	7.74		13.5		
			K				32	106		30		
		$\bar{X}$	E(C)				.7820	2.065		.8460		
			N				2	3		2		
			L				3.50	1.75		2.75		
			K				26	105		30		
	100	p	E(C)	.4948	.8026	1.7757	.9029	2.2267	5.8864	.9569	2.4460	6.7358
			N	2	3	11	2	5	17	2	2	6
			L_{max}	13.5	11.0	5.64	13.5	8.5	4.46	13.5	13.5	7.74
			K	19	42	148	33	106	425	31	92	355
		$\bar{X}$	E(C)	.460	.737	1.662	.875	2.205	5.830	.941	2.430	6.789
			N	2	3	8	2	3	9	2	2	2
			L	3.00	2.75	2.25	3.00	3.75	2.00	3.50	3.25	3.75
			K	20	46	150	34	85	420	30	85	290
	1000	p	E(C)				1.9061	3.1432	6.6491			
			N				2	3	10			
			L_{max}				13.5	11.0	5.93			
			K				40	110	450			
		$\bar{X}$	E(C)				1.784	3.044	6.571			
			N				3	4	12			
			L				3.75	4.00	3.00			
			K				36	95	440			

The optimal results of the $\overline{X}$ chart model were obtained through the use of a grid search technique. The size of the grid employed is unknown. The optimal values of K are accurate to only two significant digits. Thus, for this example with K = 290, the grid size for K could have been as large as ± 10.

The optimal solutions for the p chart model were obtained with an accuracy of ± 1 for both N and K. If this apparent difference in the accuracy between models exists, then the optimal solution for the $\overline{X}$ chart model with a smaller grid size for K should result in an expected total cost less than that for the p chart model.

The effect on the optimal solution of changing a_1 and π can be shown by comparing the difference in the minimum expected costs of the two models as the values of the cost a_1 and the parameter π are varied.

When a_1 is small relative to the other costs, increasing the sample size increases the total cost of sampling at a faster rate than when a_1 is large. An equal sample size in both models with $a_1 = 10$ indicates that the $\overline{X}$ chart model has the economic advantage of a more powerful test to detect a given shift in the process. Thus, the cost of rejecting H_0 in the $\overline{X}$ chart is less than in the p chart model. As the optimal value of K in both charts is similar, the expected cost of producing defectives before a test is performed is approximately the same in both cases. The economic advantage of the $\overline{X}$ chart is greater when π is small, because a more powerful test is required to detect small shifts in the process. Therefore, the difference in the total expected cost between the two models must result from the difference between the expected cost of rejecting H_0 in the two models.

A brief examination of the results shown in Table 2 indicates changing the value of either a_1 or π has the greatest effect on the optimal solutions. The fact that K and N increase and L decreases when a_1 increases shows less frequent, more powerful test to be more economical. When K increases, the expected cost of producing defectives before a test is performed increases. When N, K, and the power of the test increase, the expected cost of not detecting a shift in the process decreases, but the type I error increases and the cost of unnecessary investigation of the process increases.

When a_1 increases with π small, the optimal sample size increases in both cases. However, the rate at which N and the power to detect a given shift increases more rapidly for the p chart than for the $\overline{X}$ chart. Thus, the cost of rejecting H_0 converges in the two models as a_1 increases. However, the larger sample size for the p chart results in a larger total sampling cost than in the $\overline{X}$ chart.

As a_1 increases, samples are taken less frequently in both cases. However, the value of K is similar for both models and the expected cost of producing defectives before a test is performed is approximately the same for both charts.

When π increases with a_1 small, the optimal values of N for both charts are similar. Thus, the total expected cost of sampling and the expected cost of producing defectives before a sample is taken are approximately equal for both charts. As the mean shift in the process increases, the difference between the expected cost of rejecting H_0 in both models becomes smaller.

When π increases, the optimal sample size decreases and L increases for both charts. With a_1 large and a_2 relatively small, the sample size does not greatly increase the total sampling cost. However, the fixed cost per sample accounts for a larger percentage of the total sampling cost. The optimal sample size for the p chart as π increases becomes progressively larger than for the $\bar{X}$ chart, but the difference in the total cost of sampling between the two models becomes smaller. As π increases, the interval between samples increases for both charts; however, the optimal value of K is larger for the p chart than for the $\bar{X}$ chart.

Changes in the value of a_3 have an effect on the optimal values of K for both charts. This is consistent as both models employ the same $\underline{p}$ and $\underline{c}$ vectors. Therefore, both models deteriorate from the in control state in the same way and the expected cost of producing defectives before a test is performed is approximately the same in both models for equal values of K. The values of K in both models are not exactly equal, as the expected sampling cost and the expected cost of rejecting H_0 are different for both models.

With a_1 and a_3 small, the optimal sample sizes are approximately equal for both charts, but as a_1 increases, the relative sample size of the two models depends on the value of a_3. With a_3 small, N for the p chart becomes increasingly larger than the optimal value of N in the $\bar{X}$ chart. However, with a_3 large the relative optimal sample size between charts reverses.

This change in the relative size of N is a result of a more rapid decrease in the economic benefit derived from sampling in the p chart than in the $\bar{X}$ chart. As K increases, the cost of producing defectives prior to sampling becomes a more significant portion of the total expected cost of producing defectives, and the expected cost of failing to reject

H_0 given a shift becomes smaller. With larger values of a_3, it also becomes more important to decrease the cost of unnecessary investigation of the process. This results in the economic feasibility of a less powerful test, which causes N to decrease and L to increase in the p chart. However, it should be noticed that as a_3 increases N decreases in the p chart and N increases in the $\bar{X}$ chart.

The variations in L_{max} in the p chart and L in the $\bar{X}$ chart are not in terms of the same units, thus little can be said about the relative magnitude of these parameters. It should be noticed that the values of L for both models react in similar ways to changes in the cost coefficients.

While the expected cost associated with the p chart is generally larger than the $\bar{X}$ chart with all model cost coefficients and parameters equal, the value of a_2 for an attribute sampling plan will, in many cases, be less than for a measurement sampling plan. Let us assume a situation where the quality of a product can be evaluated either by measuring the value of some product characteristic or by comparing the same characteristic on the basis of a standard. Under these circumstances, one may obtain cheaper observations by an attribute inspection procedure. This can be reflected by reducing the variable sampling cost, a_2. If a_2 is decreased to 30 percent of its original value and if the comparison is repeated with one of the examples in Table 2, then the effect on the optimal solution can be evaluated.

Consider the case were $a_1 = 10$, $a_2 = 0.1$, $a_3 = 100$, $a_4 = 10$, $\pi = .597$, $S = 6$, and $\lambda' = 1000$. The results of this comparison are shown in Table 3.

Table 3. The Effect of Decreasing a_2 on the p Chart Model

	$\bar{X}$ chart $a_2 = 1.0$	p chart $a_2 = 1.0$	p chart $a_2 = 0.3$
E(C)	.875	.9029	.8587
N	2	2	3
L	3.0	13.5 (L_{max})	11.0 (L_{max})
K	34	33	32

By decreasing a_2, the expected cost for the p chart is less than the cost for the original $\bar{X}$ chart. Also, the optimal sample size increases, and L decreases. These results, while not dramatic, are in the expected direction. That is, the effect on the optimal solution is a larger sample size with a more powerful test. The net effect of these changes can be shown for $a_2 = 1.0$: $\underline{q} = (.0054, .0451, .2922, .7500, .9748, .9995, 1.0000)$,

and $E(C_1) = .364$, $E(C_2) = .113$, $E(C_3) = .426$, and

for $a_2 = 0.3$: $\underline{q} = (.0081, .0669, .4045, .8750, .9960, 1.0000, 1.0000)$,

and $E(C_1) = .341$, $E(C_2) = .122$, $E(C_3) = .396$.

3.2 The Effect of Changing the $\underline{p}$ Vector

The sensitivity of the optimal values of the test parameters for the attribute sampling plan to changes in the definition of the vector $\underline{p}$ was investigated and the results are shown in Table 4.

The model parameters chosen were $a_3 = 100$, $\lambda' = 1000$, $S = 6$, and $\pi = .597$, and the remaining coefficients (a_1, a_2, and a_4) were varied over a range of values. To evaluate the effect on the optimal solution of

Table 4. Optimal Test Parameters N, L, K and the Minimum Expected Cost Per Unit E(C) as a Function of Three Fractions Defective Vectors and a_1, a_2, and a_4 with a_3=100, λ'=1000, π=.597, and S=6. Optimal Control Procedure Is to Reject H_0 When $D \geq 1$.

a_1	a_2	a_4	Parameters	Fraction Defective Vectors $\underline{p}_1$	$\underline{p}_2$	$\underline{p}_N$
10	1.0	10	E(C)	.4715	.7480	.9029
			N	10	4	2
			L_{max}	2.86	4.82	13.5
			K	260	68	33
		20	E(C)	.7392	1.0967	1.2583
			N	10	4	2
			L_{max}	2.86	4.824	13.5
			K	149	46	23
5	0.5	10	E(C)	.4255	.6313	.6897
			N	8	4	2
			L_{max}	3.27	4.82	13.5
			K	164	52	23
		20	E(C)	.6614	.9255	.9540
			N	8	4	2
			L_{max}	3.27	4.82	13.5
			K	98	36	16
1.0	0.1	10	E(C)	.3699	.4835	.4184
			N	6	2	2
			L_{max}	3.86	6.97	13.5
			K	89	23	12
		20	E(C)	.5708	.7099	.5682
			N	6	2	2
			L_{max}	3.86	6.97	13.5
			K	56	16	8

decreasing the expected cost of sampling, the values of a_1 and a_2 were decreased to 50 percent and then to 10 percent of their respective original values. To evaluate the effect on the optimal solution of increasing the expected cost of producing defectives, the value of a_4 was increased 100 percent.

Each production process has a unique fractions defective vector and a unique deterioration rate from which the vectors $\underline{p}$ and $\underline{c}$ must be computed to accurately model the shifts particular to the process in question. While a single $\underline{p}$ vector cannot be used to model the fractions defective for all production process, it was felt that $\underline{p}_N$ defined shifts in the process fraction defective larger than normally associated with an attribute sampling plan. Therefore, to evaluate the effect of changing $\underline{p}$ on the optimal solution, and realizing an attribute sampling plan is usually associated with smaller shifts in the fraction defective, two additional $\underline{p}$ vectors were defined as follows:

$$\underline{p}_1 = (.01, .02, .03, .04, .05, .06, .07),$$

and

$$\underline{p}_2 = (.01, .02, .04, .08, .16, .32, .64).$$

The effect on the optimal solution of a decrease in a_1 and a_2 produced smaller sample sizes and intervals between samples, with wider control limits. This is to be expected, as it is economical to take less powerful, more frequent tests. When a_4 is increased, the interval between samples is decreased. This is consistent with the relatively greater cost per defective. When K decreases, fewer units are produced between samples before it is economically feasible to sample and test the hypotheses.

The effect on the optimal solution of a change in the vector $\underline{p}$

indicates that in order to detect smaller shifts in the process fraction defective, the optimal solution yields larger values of N and smaller values of L. This results in a more powerful test to detect the smaller shifts. With smaller shifts in the fraction defective, the process deteriorates slower in the sense that fewer defectives are produced. When fewer defectives are produced the expected cost associated with their production decreases, and the optimal value of K increases.

3.3 Experimental Results

The results in this section, shown in Tables 5, 6, and 7, are based on the fractions defective vector $\underline{p}$ = (.01, .02, .04, .08, .16, .32, .64) for several values of the cost coefficients (a_1, a_2, a_3, and a_4) the previous values of the parameter π (that is, π = .376, .597, .800) and S = 6 out of control states.

The range of the cost coefficients is limited by their relative magnitude. If a_3 is large relative to a_4, then at some point in becomes uneconomical to investigate and correct the process for assignable causes. When this occurs, the system continues to operate indefinitely as the cost of producing defectives never justifies the cost of investigating and correcting the process. This occurred several times when a_3 = 1000, and these cases are denoted in Tables 5 and 6 by $K = \infty$. If a_3 is too small relative to a_1 and a_2, then at some point it becomes uneconomical to sample. When this occurs, N = 2 (lower limit) and L = 0. The process, regardless of the sample statistic $\hat{p}$, will always be investigated and corrected for assignable causes after the production of K units. This occurred several times when a_3 = 5 or a_3 = 10, and these cases are denoted in Tables 5 and 6 by L = 0.

Table 5. Optimal Test Parameters N, L, and K and Minimum Expected Cost Per Unit E(C) as a Function of a_1, a_2, and a_3 with π=.376, a_4=10, λ'=1000, S=6, and $p=p_2$. Optimal Control Procedure Is to Reject H_0 When D≥1.

a_2	a_3	Parameters	π=.376 a_1 0.5	1.0	2.5	5.0	10.0
0.05	5	E(C)	.2088	.2262	.2601	.2976	.3499
		N	7	10	17	24	35
		L_{max}	3.53	2.86	2.02	1.56	1.10
		K	23	33	55	78	113
	50	E(C)	.3238	.3392	.3703	.4049	.4527
		N	4	5	7	10	16
		L_{max}	4.82	4.27	3.53	2.86	2.11
		K	31	40	58	83	126
	250	E(C)	.6232	.6351	.6677	.6850	.7197
		N	2	2	2	4	7
		L_{max}	6.97	6.97	6.97	4.82	3.53
		K	41	43	49	93	155
0.10	10	E(C)	.2372	.2546	.2881	.3251	.3765
		N	5	7	12	17	24
		L_{max}	4.27	3.53	2.55	2.02	1.56
		K	24	34	56	80	115
	20	E(C)	.2643	.2807	.3136	.3498	.4000
		N	5	6	10	14	20
		L_{max}	4.27	3.86	2.86	2.31	1.80
		K	28	35	57	81	117
	100	E(C)	.4212	.4323	.4611	.4931	.5380
		N	4	4	6	7	10
		L_{max}	4.82	4.82	3.86	3.53	2.86
		K	44	46	70	88	128
	250	E(C)	.6257	.6375	.6697	.6871	.7219
		N	2	2	2	4	7
		L_{max}	6.97	6.97	6.97	4.82	3.53
		K	41	43	50	94	156
	1000	E(C)					
		N					
		L_{max}					
		K	∞	∞	∞	∞	∞
0.25	5	E(C)	.2528	.2695	.3037	.3408	
		N	5	6	10	13	
		L_{max}	4.27	3.86	2.86	2.43	0.0
		K	28	35	58	79	
	50	E(C)	.3478	.3612	.3926	.4272	.4751
		N	4	4	6	9	12
		L_{max}	4.82	4.82	3.86	3.05	2.55
		K	36	39	59	87	122
	250	E(C)	.6328	.6443	.6757	.6917	.7294
		N	2	2	2	5	6
		L_{max}	6.97	6.97	6.97	4.27	3.86
		K	43	45	51	112	146

Table 6. Optimal Test Parameters N, L, and K and Minimum Expected Cost Per Unit E(C) as a Function of a_1, a_2, and a_3 with π=.376, a_4=10, λ'=1000, S=6, and $p=p_2$. Optimal Control Procedure Is to Reject H_0 When $D \geq 1$.

			π=.376				
			a_1				
a_2	a_3	Parameters	0.5	1.0	2.5	5.0	10.0
0.50	5	E(C)					
		N					
		L_{max}	0.0	0.0	0.0	0.0	0.0
		K					
	10	E(C)	.2968	.3140	.3476	.3842	
		N	3	4	6	9	
		L_{max}	5.63	4.82	3.86	3.05	0.0
		K	26	35	54	80	
	20	E(C)	.3189	.3329	.3657	.4019	.4514
		N	4	4	6	8	12
		L_{max}	4.82	4.82	3.86	3.27	2.55
		K	34	37	57	80	119
	50	E(C)	.3698	.3845	.4166	.4504	.4980
		N	3	3	6	7	10
		L_{max}	5.63	5.63	3.86	3.53	2.86
		K	33	35	66	84	122
	100	E(C)	.4547	.4642	.4908	.5224	.5670
		N	4	4	5	6	9
		L_{max}	4.82	4.82	4.27	3.86	3.05
		K	52	54	71	90	135
	250	E(C)	.6443	.6552	.6854	.7036	.7394
		N	2	2	2	4	6
		L_{max}	6.97	6.97	6.97	4.82	3.86
		K	43	47	53	100	152
	1000	E(C)					
		N					
		L_{max}					
		K	∞	∞	∞	∞	∞
1.00	10	E(C)	.3475	.3643	.3951		
		N	3	4	5		
		L_{max}	5.63	4.82	4.27	0.0	0.0
		K	34	45	60		
	20	E(C)	.3640	.3777	.4105	.4458	.4944
		N	3	3	5	6	9
		L_{max}	5.63	5.63	4.27	3.86	3.05
		K	35	38	62	81	122
	100	E(C)	.4835	.4985	.5214	.5536	.5970
		N	3	4	4	6	7
		L_{max}	5.63	4.82	4.82	3.86	3.53
		K	47	63	69	102	131
	250	E(C)	.6657	.6757	.7037	.7229	.7585
		N	2	2	2	4	6
		L_{max}	6.97	6.97	6.97	4.82	3.86
		K	49	51	57	108	163
	1000	E(C)					
		N					
		L_{max}					
		K	∞	∞	∞	∞	∞

Table 7. Optimal Test Parameters N, L, and K and Minimum Expected Cost Per Unit E(C) as a Function of a_1, a_2, and a_3 with π=.597, π=.800, a_4=10, λ'=1000, S=6, and $p=p_2$. Optimal Control Procedure Is to Reject H_0 When D≥1.

a_2	a_3	Parameters	π =.597			π =.800		
			a_1			a_1		
			1.0	5.0	10.0	1.0	5.0	10.0
0.1	10	E(C)	.2939	.4132	.5009	.3276	.5000	.6287
		N	4	10	14	3	6	8
		L_{max}	4.82	2.86	2.31	5.63	3.86	3.27
		K	20	47	67	15	32	45
	100	E(C)	.4835	.5992	.6833	.5247	.6858	.8077
		N	2	4	6	2	2	4
		L_{max}	6.97	4.82	3.86	6.97	6.97	4.82
		K	23	48	70	20	30	48
	1000	E(C)	1.6335	1.6818	1.7358	1.8328	1.8994	1.9749
		N	2	2	2	2	2	2
		L_{max}	6.97	6.97	6.97	6.97	6.97	6.97
		K	79	87	98	57	63	70
0.5	10	E(C)	.3542	.4743	.5614	.3846	.5553	.6819
		N	2	6	8	2	4	5
		L_{max}	6.97	3.86	3.27	6.97	4.82	4.27
		K	18	49	68	16	34	46
	100	E(C)	.5167	.6313	.7148	.5630	.7118	.8364
		N	2	4	5	2	2	3
		L_{max}	6.97	4.82	4.27	6.97	6.97	5.63
		K	26	52	70	22	32	46
	1000	E(C)	1.6436	1.6908	1.7439	1.8467	1.9120	1.9863
		N	2	2	2	2	2	2
		L_{max}	6.97	6.97	6.97	6.97	6.97	6.97
		K	80	89	99	58	64	71
1.0	10	E(C)	.4033	.5231	.6104	.4406	.6025	.7271
		N	2	4	6	2	2	4
		L_{max}	6.97	4.82	3.86	6.97	6.97	4.82
		K	22	47	69	20	30	48
	100	E(C)	.5535	.6633	.7480	.6056	.7423	.8679
		N	2	3	4	2	2	3
		L_{max}	6.97	5.63	4.82	6.97	6.97	5.63
		K	29	49	68	25	34	49
	1000	E(C)	1.6559	1.7019	1.7539	1.8635	1.9274	2.0003
		N	2	2	2	2	2	2
		L_{max}	6.97	6.97	6.97	6.97	6.97	6.97
		K	83	91	101	60	65	72

The previous values of π were used in this phase of the investigation. However, the expected magnitude of a shift in the process fraction defective given that a shift occurs depends upon $\underline{p}$ and is not the same as before. The parameter π still skews the a priori probability distribution $\underline{c}$ over the out of control states by changing the relative magnitudes of the probabilities c_i, $i = 1, 2, \ldots, S$. For $\pi = .376$ the skewness of c_i, $i = 1, 2, \ldots, S$, is such that c_2 and c_3 will be the largest probabilities or the process will most likely shift from the in control state to either state 2 or 3. The expected value of this shift represents a relatively small deterioration in the process fraction defective. For $\pi = .597$, c_3 and c_4 will be the largest probabilities and the expected value of the shift represents a moderate deterioration in the process fraction defective. For $\pi = .800$, c_4 and c_5 will be the largest probabilities, and the expected value of the shift represents a relatively large deterioration in the process fraction defective.

Six out of control states were used in the study. This is considered to be the upper limit on the number of unique out of control states that could be effectively isolated. Knappenberger and Grandage (16) have conducted some research with their $\overline{X}$ chart model using more than six out of control states, but they found little or no effect on the optimum solution. The effect of using fewer than six out of control states for the p chart will be discussed in section 3.5.5.

The optimum control procedure for the results shown in Tables 5, 6, 7, and 8, is to take a sample of size N every K units produced and to reject H_0 if $D \geq 1$. The values of L_{max} and L_{min} and of the possible criteria for rejection with $\underline{p}_2$ and $2 \leq N \leq 30$ are listed in Appendix B. Other criteria for rejection are possible with different cost coefficients and parameters. The effect of changing the fractions defective vector on the criteria for rejection is shown in Table 11.

3.4 Significant Effects and Interactions

To analyze the results shown in Tables 5, 6, and 7, it was necessary to determine which variables (a_1, a_2, a_3, a_4, and π) have a significant effect on the optimal test parameters. Considerable difficulty was experienced in determining the magnitude of the effect for each variable and in determining the effect of the interactions between these variables on the optimal solution.

A fractional factorial experiment was designed to study the effect of these variables and to determine the existence of significant interactions between these variables on the optimal solution of the p chart model with $\underline{p} = \underline{p}_2$, S = 6, and $\lambda' = 1000$. The design consisted of a 2^{5-1} fractional factorial experiment with the defining contrast I = 12345.

By utilizing this defining contrast, all main effects are aliased with fourth-order interactions, and all two factor interactions are aliased with third-order interactions. However, no main effect or two factor interaction is aliased with any other main effect or two factor interaction. This is a practical design as there are good tests on all main effects and two factor interactions, assuming all higher-order interactions are zero.

This design consisted of the five variables taken at two levels, where the minus version corresponds to the lower value and the plus version to the upper value of the variables. The values of the variables at the two levels for this experiment are listed following.

Variable		Version (-)	Version (+)
(1)	a_1	2.5	10.0
(2)	a_2	0.1	1.0
(3)	a_3	50.0	250.0
(4)	a_4	10.0	20.0
(5)	π	.376	.800

For each of the sixteen experiments conducted, the response consisted of the optimal sample size, interval between samples, and precentage of units produced which are inspected. The significant main effects and two factor interactions found by the usual analysis are listed in Table 8 by descending order of magnitude for each response.

Table 8. Significant Main Effects and Two Factor Interactions Listed in Descending Order of Magnitude

Optimal Sample Size (N)		Interval Between Samples (K)		Percentage of Units Inspected	
Variable	Effect	Variable	Effect	Variable	Effect
5	-3.875	5	-51.125	3	-0.391
1	+2.625	1	+37.875	4	+0.335
3	-2.375	15	-22.875	23	+0.201
15	-1.625	4	-21.125	2	-0.195
23	+1.625	14	-14.875		
35	+1.375	23	+12.375		
2	-1.125	45	-10.625		

The estimates listed above are suspiciously large when compared to the others. The simplest interpretation of the results would be that the above main effects and interactions are important.

The most likely explanation of the results, shown in Table 8, indicates that the variables a_1 and π have a significant effect on the optimal values of both N and K. The variables a_4 and a_3 having a less significant effect on the optimal values of N and K respectively. The variables a_3, a_4, and a_2, have a significant effect on the optimal percentage of units inspected.

The significant interactions listed in Table 8 are further interpreted in Figure 4. A brief examination of the graphs indicates that most of the lines are nearly parallel. This represents a characteristic of small interactions between the two factors. This shows that an increase of one variable produced about the same average increase in the response regardless of the level of the other variable.

The fact that the graphs in Figure 4 for interactions containing either the variable a_1 or π indicate a response quite different for a_1 and π at different levels shows the variables a_1 and π have a significant effect on the optimal values of both N and K.

In some cases, Figure 4 indicates significant interactions which, in reality, may not exist. Therefore, conclusions drawn from these results must be qualified by the fact that the optimal sample size has a lower limit ($N \geq 2$) which, under certain changes in the variables, has an effect on the optimal values of N, K, and the percentage of units sampled.

For example, the interaction a_2a_3, shows the effect of increasing a_2 on the percentage of units inspected changing according to the relative level of a_3. The rate of decrease in the percentage of units inspected, as a_2 increases, is greater with a_3 at the low level, but in this case

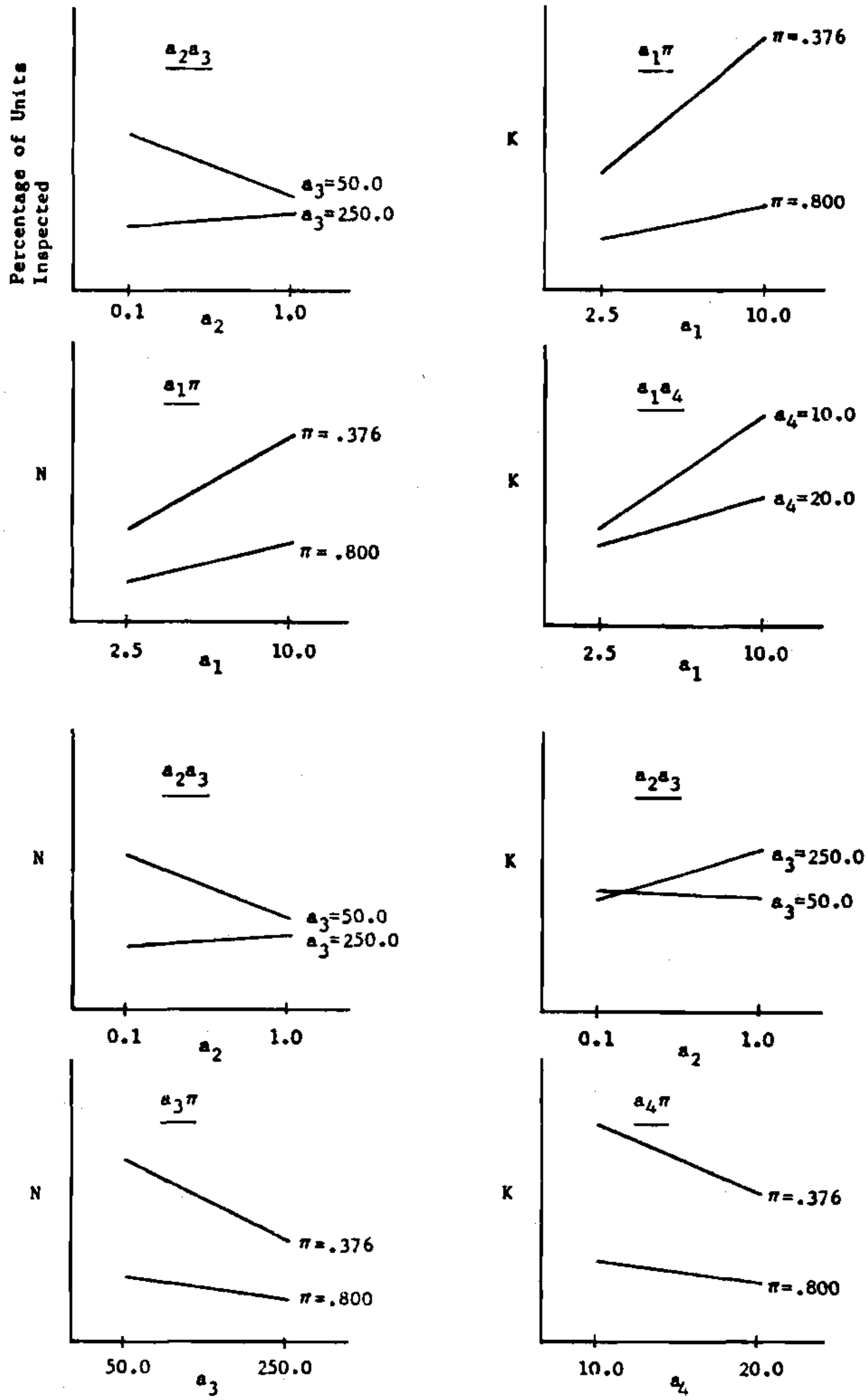


Figure 4. Significant Interactions for the Optimal Values of N, K, and the Percentage of Units Inspected

$N > 2$ and an increase in a_2 can cause a decrease in N which accounts for a decrease in the percentage of units inspected. With a_3 at the high level, $N \geq 2$, and an increase in a_2 cannot always cause a decrease in N. If $N = 2$, the result of increasing a_2 causes a decrease in K which accounts for the slight increase in the percentage of units inspected.

A similar argument can be used when explaining the apparent change in the effect of increasing a_2 has on N and K for different values of a_3. Thus, without the lower limit on N ($N \geq 2$), the effect of increasing a_2 on N, K, and the percentage of units inspected with a_3 at the high level would be the same as with a_3 at low level.

The lower limit on N ($N \geq 2$) also affects the interactions involving the variable π. Thus, the effect on N and K of an interaction involving π at the high level would be similar to the effect on N and K of the same interaction involving π at the low level.

3.5 Sensitivity Analysis

The purpose of this section is to analyze the behavior of the optimal test parameters (N, L, K) to changes in the cost coefficients (a_1, a_2, a_3, and a_4), the mean shift of the process given a shift occurs (π), the number of out of control states (S), the mean deterioration rate of the process (λ'), and the fractions defective, p_i, $i = 0, 1, 2, . . ., S$.

3.5.1 Sensitivity to Changes in the Cost Coefficients

To determine the behavior of the optimal test parameters to changes in the cost coefficients (a_1, a_2, a_3, and a_4), the experimental results shown in Tables 5, 6, and 7 were analyzed according to the significant main effects and interactions found in section 3.4.

By increasing the fixed cost per sample, a_1, the sample size and the interval between samples increase, but the width of the control limits decreases. This is to be expected, as it is economical to take more powerful, less frequent tests. The rate at which N increases as a_1 increases depends upon the relative magnitude of π. For small values of π, N increases at a faster rate than for larger values of π.

K increases because the cost of producing defectives increases to a point where the higher cost of sampling is justified. The amount K increases depends upon the relative magnitude of π and a_4. For small values of a_4 and π, K will increase at a faster rate or the cost of producing defectives will increase at a slower rate than for larger values of either a_4 or π.

By increasing the variable cost of sampling, a_2, the sample size decreases and the width of the control limits increases. As the cost to take an observation increases, the power of the test decreases. The rate at which N decreases as a_2 increases depends upon the relative magnitude of a_3. For small values of a_3, N decreases, but for large values of a_3, N is not affected.

The change in K as a_2 increases depends upon the relative size of a_3. K increases for large values of a_3, but for smaller values of a_3, K is not affected.

The actual changes in N and K under these circumstances determine the percentage of units produced which is inspected. For small values of a_3 the percentage of units inspected decreases as a_2 increases, but for larger values of a_3, the percentage increases slightly. In general, as a_2 increases, the percentage of units inspected decreases.

By increasing the cost of investigating and correcting the process,

a_3, the percentage of units produced which is inspected decreases. This usually results from either decreasing N or increasing K, or both. As a general rule, K increases because it is economical to produce more defectives before the process is investigated and corrected for assignable errors. However, a change in the sample size, as a_3 increases, depends upon the relative magnitude of π. When π is large, the sample size will slightly decrease, but when π is small, the sample size decreases at a faster rate.

By increasing the cost per defective, a_4, the interval between samples decreases because it is economical to detect assignable errors in the process more quickly. This usually results in a larger percentage of units produced to be inspected. As a_4 increases, the rate at which K decreases depends upon the relative magnitude of a_1. For large values of a_1, K decreases at a faster rate than for smaller values of a_1.

3.5.2 Sensitivity to Changes in the Mean Shift of the Process Given a Shift Occurs (π)

To illustrate the effect on the optimal solution of a change in the mean shift of the process given a shift occurs, three values of the *a priori* distribution parameter π were used. By increasing the mean shift of the process, the sample size decreases, the width of the control limits increases (L_{max} increases), and the interval between samples decreases. This is to be expected as a less powerful test becomes adequate to detect the larger process shifts. The interval between samples decreases because when the process does go out of control, it will on the average produce more defectives. This increases the expected cost of producing defectives. By decreasing K, the expected sampling cost increases, but this cost increase is traded off with the economic advantage of detecting a shift in the process more quickly.

The rate at which K decreases as π increases depends upon the relative magnitude of a_1 and a_4. For small values of a_1, K decreases at a slower rate than for larger values of a_1. This is due to the fact that K has already been decreased by the economic advantage of taking frequent samples when a_1 is small. For small values of a_4, K decreases at a slower rate than for larger values of a_4. This is a result of the rapid increase in the expected cost of producing defectives when either a_4 or π increases, and the economic advantage of taking frequent samples when both a_4 and π are large.

3.5.3 Sensitivity to the Number of Out of Control States

To determine the effect on the optimal solution of a change in the number of out of control states (S), several modifications to the model were necessary. The model developed in Chapter II and the computer program shown in Appendix A are capable of optimizing a process with $S \geq 3$, but for S = 1 or 2, they had to be modified. The results of this analysis are shown in Table 9.

To compare the optimal solutions of a process with S = 6 and with S = 1 or 2, a new fraction defective vector $\underline{p}$ was developed from $\underline{p}_2$. From Table 5, the experimental run with parameters $a_1 = 5.0$, $a_2 = 20.0$, $a_4 = 10.0$, $\lambda' = 1000$, and $\pi = .376$ was used. The starting argument for the Hooke and Jeeves pattern search in this example was $\underline{X} = (10, 1.5, 60)$; therefore, the models with S = 1 and S = 2 were started from the same initial values of the test parameters.

With S = 6, $\underline{X} = (10, 1.5, 60)$ and $\underline{p} = (.01, .02, .04, .08, .16, .32, .64)$, the model generated the following value for the vector $\underline{c}$: $\underline{c}$ = (.9418, .0132, .0199, .0160, .0072, .0017, .0002).

The method used to define the new $\underline{p}$ was to let the in control fractions defective remain the same, but the new out of control fractions defective were computed to represent a weighted average of p_i, $i = 1, 2, \ldots, 6$, in $\underline{p}_2$.

For S = 1 define

$$p_0(S = 1) = 0.01 \quad \text{and}$$

$$p_1(S = 1) = \sum_{i=1}^{6} c_i p_i \Big/ \sum_{i=1}^{6} c_i = 0.004164/0.0582 = 0.0716$$

Thus, $\underline{p}(S = 1) = (0.01, 0.0716)$.

The following terms were redefined:

(1) The vector $\underline{c} = (c_0, c_1) = (e^{-\lambda' K}, 1 - e^{-\lambda' K})$

(2) The transition matrix B,

$$B = \begin{bmatrix} c_0 & c_1 \\ & \\ q_1 p_0 & q_1 p_1 + (1 - q_1) \end{bmatrix}$$

(3) The vector $\underline{\gamma} = (\gamma_0, \gamma_1)$, where

$$\gamma_0 = \alpha_0 c_0 + F\alpha_0 c_1 \quad \text{and}$$

$$\gamma_1 = \alpha_1 + (1 - F)\alpha_0 c_1 .$$

For S = 2 define

$$p_0(S = 2) = 0.01 \ ,$$

$$p_1(S = 2) = \sum_{i=1}^{3} c_i p_i \Big/ \sum_{i=1}^{3} c_i = 0.00234/0.0491 = 0.0477, \text{ and}$$

$$p_2(S = 2) = \sum_{i=4}^{6} c_i p_i \Big/ \sum_{i=4}^{6} c_i = 0.001824/0.0091 = 0.2004 \ .$$

Thus, $\underline{p}(S = 2) = (0.01, 0.0477, 0.2004)$.

The following terms were redefined:

(1) The vector $\underline{c} = (c_0, c_1, c_2)$, where

$$c_0 = e^{-\lambda' K} \ ,$$

$$c_1 = \frac{2(1 - e^{-\lambda' K})}{(1 - (1 - \pi)^2)} \pi(1 - \pi) \ , \text{ and}$$

$$c_2 = \frac{(1 - e^{-\lambda' K})\pi^2}{(1 - (1 - \pi)^2)} \ .$$

(2) The transition matrix B,

$$B = \begin{bmatrix} c_0 & c_1 & c_2 \\ q_1 c_0 & q_1 c_1 + \dfrac{(1 - q_1)c_1}{(1 - c_0)} & q_1 c_2 + \dfrac{(1 - q_2)c_2}{(1 - c_0)} \\ q_2 c_0 & q_2 c_1 & \dfrac{(1 - q_2)(c_1 + c_2)}{(1 - c_0)} \end{bmatrix}$$

(3) The vector $\underline{\gamma} = (\gamma_0, \gamma_1, \gamma_2)$, where

$$\gamma_0 = \alpha_0 c_0 + F\alpha_0(1 - c_0) ,$$

$$\gamma_1 = \frac{\alpha_1 c_1}{(1 - c_0)} + \alpha_0(1 - F)c_1 + \frac{\alpha_1 F c_2}{(1 - c_0)} , \text{ and}$$

$$\gamma_2 = \frac{\alpha_2(c_1 + c_2)}{(1 - c_0)} + \alpha_0(1 - F)c_2 + \frac{c_2(1 - F)\alpha_1}{(1 - c_0)} .$$

The results of optimizing the models with S = 1 and S = 2 out of control states are shown in Table 9. The models (S = 1, 2) based on the vectors $\underline{p}$, $\underline{c}$, and $\underline{\gamma}$ defined in this section produced the minimum expected cost, $E(C)^*$, and the values of N, L, and K shown. If we assume the model with S = 6 out of control states accurately models the process, then by substituting these values of N, L, and K into the original model the effect of changing S can be determined.

The expected cost for the sampling plans based on the models with S = 1 and S = 2 out of control states were recalculated by the original model (S = 6). These expected costs, denoted by E(C), were compared to E(C) of the sampling plan based on the model with S = 6. The comparison is in terms of percent error.

Table 9. The Effect of Changing S on the Optimal Solution of a Process with $a_1 = 5.0$, $a_2 = 0.1$, $a_3 = 20.0$, $a_4 = 10.0$, $\lambda' = 1000$, $\pi = .376$, and $\underline{p} = \underline{p}_2$

	S = 1	S = 2	S = 6
E(C)*	\$0.3163	\$0.3583	-------
E(C)	\$0.3525	\$0.3500	\$0.3498
Error (%)	0.71%	0.057%	-------
N	14	14	14
L_{max}	2.31	2.31	2.31
K	96	78	81

While models with one or two out of control states do not greatly increase the expected cost for this set of model cost coefficients and parameters, this always may not be the case. The change in E(C) for these cost coefficients is relatively insensitive to changes in K, but if a_3 or a_4 were larger, then variations in K may become more critical.

The model seems relatively insensitive to the number of out of control states, but the effect of changing S is very dependent upon the method used to define the fractions defective for the new out of control states. In this case the deterioration rate of the process in the models with S = 1, 2, and 6 is equal ($\lambda' = 1000$), but the expected cost of the process deteriorating from the in control state depends upon the number of out of control states, the fraction defective vector, and the probability of the process shifting to each of the out of control states.

The practical significance of choosing the optimal number of out of control states has several implications. The smaller S is, the less complicated the model becomes, but at the expense of a less accurately designed sampling plan. With the use of a computer, the complexity of the model is no longer a problem in finding an optimal solution. The computer time required to find the optimal solution is relatively insensitive to the value of S. Therefore, the number of out of control states should be kept to a minimum, but not so small as to cause significant errors in the optimal sampling plan. The actual value of S should be determined by the process being modeled and the desired accuracy in the solution.

3.5.4 Sensitivity to Increasing the Mean Deterioration Rate (λ')

To illustrate the effect on the optimal solution of a change in the mean of deterioration rate of the process, the value of λ' is increased from 1000 units between shifts to 10,000 units. The results, shown in

Table 10, are for $a_4 = 10$, $\pi = .376$, $\underline{p} = \underline{p}_2$, $S = 6$, $\lambda' = 10{,}000$, and several values of a_1, a_2, and a_3. These results were compared to similar results with $\lambda' = 1000$ in Table 5.

This comparison indicated that the test parameters N and L were slightly affected by increasing the value of λ'. This is to be expected, as the magnitude of shifts in the process fraction defective and the cost coefficients were not changed. N and L were not significantly changed because the cost to detect a given shift with an equal power remained the same.

A change in λ' greatly affected the interval between samples. For these examples, K increases two to three times in value when λ' is increased by a factor of ten. This is consistent with assignable errors in the process occurring less frequently and with the economic feasibility of producing more units between samples. The expected cost of producing defectives will remain approximately the same when λ' and K change in this manner.

3.5.5 Sensitivity to Changes in p_i, $i = 0, 1, 2, \ldots, S$

To illustrate the effect on the optimal solution of a change in the fractions defective for each state, a new fraction defective vector was defined as $\underline{p}_3 = (.20, .21, .23, .27, .35, .51, .83)$ where the fractions defective for $\underline{p}_2$ were increased by 0.19. The results shown in Table 11 are for $a_4 = 10$, $\pi = .376$, $\lambda' = 1000$, $S = 6$, $\underline{p} = \underline{p}_3$, and several values of a_1, a_2, and a_3. These results were compared to similar results with $\underline{p} = \underline{p}_2$ in Table 5.

An important difference between the results using $\underline{p}_3$ and $\underline{p}_2$ is shown in Table 11. When $\underline{p}_2$ was used, a high percentage of defectives was

Table 10. Sensitivity of the Optimal Solutions to an Increase in λ'. Optimal Control Procedure Is to Reject H_0 When $D \geq 1$.

			$a_4=10$, $\pi=.376$, $\underline{p}=\underline{p}_2$, $\lambda'=10{,}000$, S=6		
			a_1		
a_2	a_3		2.5	5.0	10.0
0.1	10.0	E(C)	.1595	.1720	.1896
		N	12	17	24
		L_{max}	2.55	2.02	1.56
		K	166	235	334
	20.0	E(C)	.1664	.1788	.1962
		N	10	14	20
		L_{max}	2.86	2.31	1.80
		K	170	238	337
	100.0	E(C)	.2030	.2156	.2323
		N	5	6	10
		L_{max}	4.26	3.86	2.86
		K	170	219	340

Table 11. Sensitivity of the Optimal Solutions to Changes in the Fractions Defective p_i, $i = 0, 1, 2, \ldots, S$

a_2	a_3	Parameters	$\underline{p}_3$=(.20,.21,.23,.27,.35,.51,.83) a_4=10, π=.376, λ'=1000, S=6		
			a_1		
			2.5	5.0	10.0
0.1	10	E(C) N L Reject H_0 if K	2.2501 10 0.8-1.58 $D=0, D\geq 4$ 50	2.2525 7 0.38-0.57 $D=0, D\geq 2$ 101	2.2795 2 0.71-1.06 $D\geq 0$* 153
	20	E(C) N L Reject H_0 if K	2.2948 10 0.8-1.58 $D=0, D\geq 4$ 60	2.3332 10 0.8-1.58 $D=0, D\geq 4$ 70	2.3375 2 0.71-1.06 $D\geq 0$* 192
	100	E(C) N L Reject H_0 if K	2.4290 38 1.78-2.19 $D\leq 2, D\geq 13$ 67	2.4656 22 1.39-1.92 $D=0, D\geq 8$ 79	2.5230 22 1.39-1.92 $D=0, D\geq 8$ 96

* Reject H_0 every K units produced

produced only when the process was out of control. Therefore, the control procedure $D \geq 1$ was determined to be optimal. With $\underline{p}_3$ the in control fraction defective equals 0.20, which is relatively larger than in the $\underline{p}_2$ vector (0.01). This increase in the fraction defective in $\underline{p}_3$ guaranteed the production of a larger percentage of defectives regardless of the state the process is operating. Table 11 shows that under $\underline{p}_3$ the optimal criteria for rejection are different.

The effect on the optimal solution of increasing the out of control fractions defective is essentially the same as increasing a_4 or increasing the mean shift of the process given a shift occurs (π). That is, the expected cost of producing defectives becomes a more significant portion of the total expected cost.

This comparison shows that by increasing the fraction defective for each state, the width of the control limits is decreased. Thus, with $\underline{p}_3$ more powerful tests are required to detect a shift in the process.

For both vectors the interval between samples increases when either a_1 or a_3 is increased. By increasing a_1 with $\underline{p}_3$, N decreases and thus the percentage of units inspected is decreased. With $\underline{p}_2$ the percentage of units inspected is not greatly affected when a_1 is increased. Thus, with $\underline{p}_3$ and large values of a_1 and small values of a_3, it becomes uneconomical to sample. With larger values of a_3, the sample size decreases as a_1 increases which shows a decrease in the economic benefit of detecting a shift in the process.

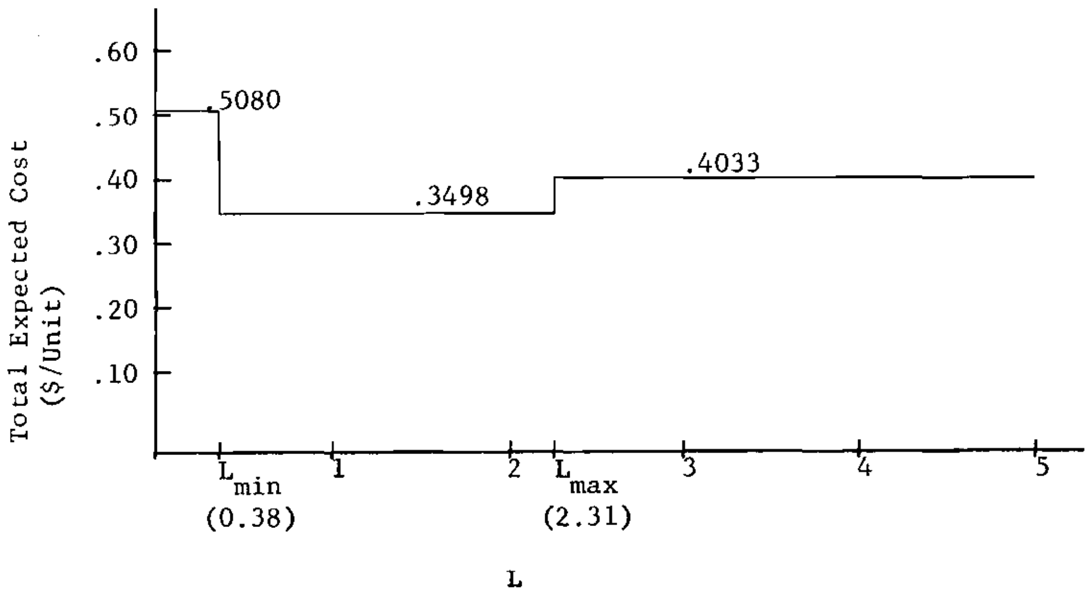
By increasing a_3 with $\underline{p}_3$, the percentage of units inspected increases, however, with $\underline{p}_2$ the percentage of units inspected is decreased. Thus, with $\underline{p}_3$ and with large values of a_3 more powerful, larger samples are economically feasible than with $\underline{p}_2$.

3.6 Behavior of the Cost Surface

The behavior of the cost surface is very important, as the global optimum can be found with the Hooke and Jeeves pattern search if the surface is convex. The objective of the analysis conducted in this section is to show the assumption of convexity in a limited region near the optimal is valid. The argument, $\underline{X} = (N, L, K)$, will be varied in different ways to show the behavior of the cost surface. The results of this analysis are shown in Figures 5, 6, 7, and 8.

The value of $L = 1.5$ is used in this section because for the fraction defective vector $\underline{p}_2$ and for the values of N and K used, the criteria for rejection in the optimal control procedure remain the same (that is, reject H_0: if $D \geq 1$). While L_{max} and L_{min} change for different values of N, the range of N will be limited such that $L = 1.5$ is always contained within the range L_{max} to L_{min}. The expected cost associated with a quality control procedure is sensitive to changes in the test parameters, but once $\underline{p}$ and N are defined, L has a range of values, L_{max} to L_{min}, within which it can vary without affecting the total cost.

The range of L which describes the optimal criteria for rejection, L_{max} to L_{min}, will produce the smallest total expected cost. When L varies outside this range, both the size of the critical region and the criteria for rejection change. Therefore, the new values of the type I and type II errors will define a new vector $\underline{q}$ used to calculate $E(C_2)$ and $E(C_3)$. As the value of L is outside the range L_{max} to L_{min}, the net change in the expected costs will increase E(C). In Figure 5, the total expected cost, E(C), is shown as a function of L and of the criteria for rejection in the control procedure.



Control Procedure:

Range of L	Criteria for Rejection
(a) $0.00 < L < 0.38$	Reject H_0 if: $D \geq 0$
(b) $0.38 < L < 2.31$	Reject H_0 if: $D \geq 1$
(c) $2.31 < L < 4.0$ (4.97)	Reject H_0 if: $D \geq 2$

Figure 5. Total Expected Cost Versus Sigma Control Limit (L) with $N = 14$, $K = 81$, $\lambda' = 1000$, $S = 6$, $\pi = .376$, $\underline{p} = \underline{p}_2$, $a_1 = 5.0$, $a_2 = 0.1$, $a_3 = 20.0$, and $a_4 = 10.0$

With $\underline{p} = \underline{p}_2$, the p chart model is shown to be relatively insensitive to large changes in L. However, the expected cost, E(C), of the model changes when the value of L defines different criteria for rejection in the control procedure. The values of L_{max}, L_{min}, and the optimal control procedure will change, depending upon the choice of $\underline{p}$ and the sample size.

To illustrate the behavior of the model cost components as the number of units produced between samples is increased, the values of $E(C_1)$

$E(C_2)$, $E(C_3)$, and $E(C)$ were calculated using K equal to 40, 50, 60, 70, 80, 90, and 100. The sample size ($N = 14$) and the control chart limits ($L = 1.5$) which define the optimal control procedure (reject H_0 if $D \geq 1$) were held constant. The results are plotted in Figure 6.

The expected sampling costs per unit, $E(C_1)$, and the expected cost per unit of rejecting the null hypothesis, $E(C_2)$, both decrease as the number of units produced between samples increase. The expected cost per unit of accepting the null hypothesis, $E(C_3)$, increases as the number of units between samples increases. The total expected cost, $E(C)$, is convex and has a minimum value at $K = 81$. Note the flatness of $E(C)$ in the vicinity of the optimum.

To illustrate the effect of changing N and K, the total expected cost was calculated for K equal to 40, 50, 60, 70, 80, 90, and 100, and N equal to 2, 3, 4, 7, 10, 13, 14, 15, 16, and 20. The control chart limits ($L = 1.5$) which define the optimal control procedure (reject H_0 if $D \geq 1$) were held constant. The minimum total cost for each N is connected with a dotted line. These results plotted in Figure 7 show the convexity of $E(C)$ as both N and K vary over limited ranges.

To further illustrate the behavior of the expected cost surface, Figure 8 plots the response surface, $E(C)$, as a function of N and K. The control chart limits ($L = 1.5$) which define the optimal control procedure (reject H_0 if $D \geq 1$) were held constant. The surface appears to be symmetric about the optimum and convex. For this example, the expected total cost, $E(C)$, is more sensitive to changes in N than to changes in K. While the convexity of the expected cost surface cannot be proven for all cases, these results tend to support the assumption of local convexity.

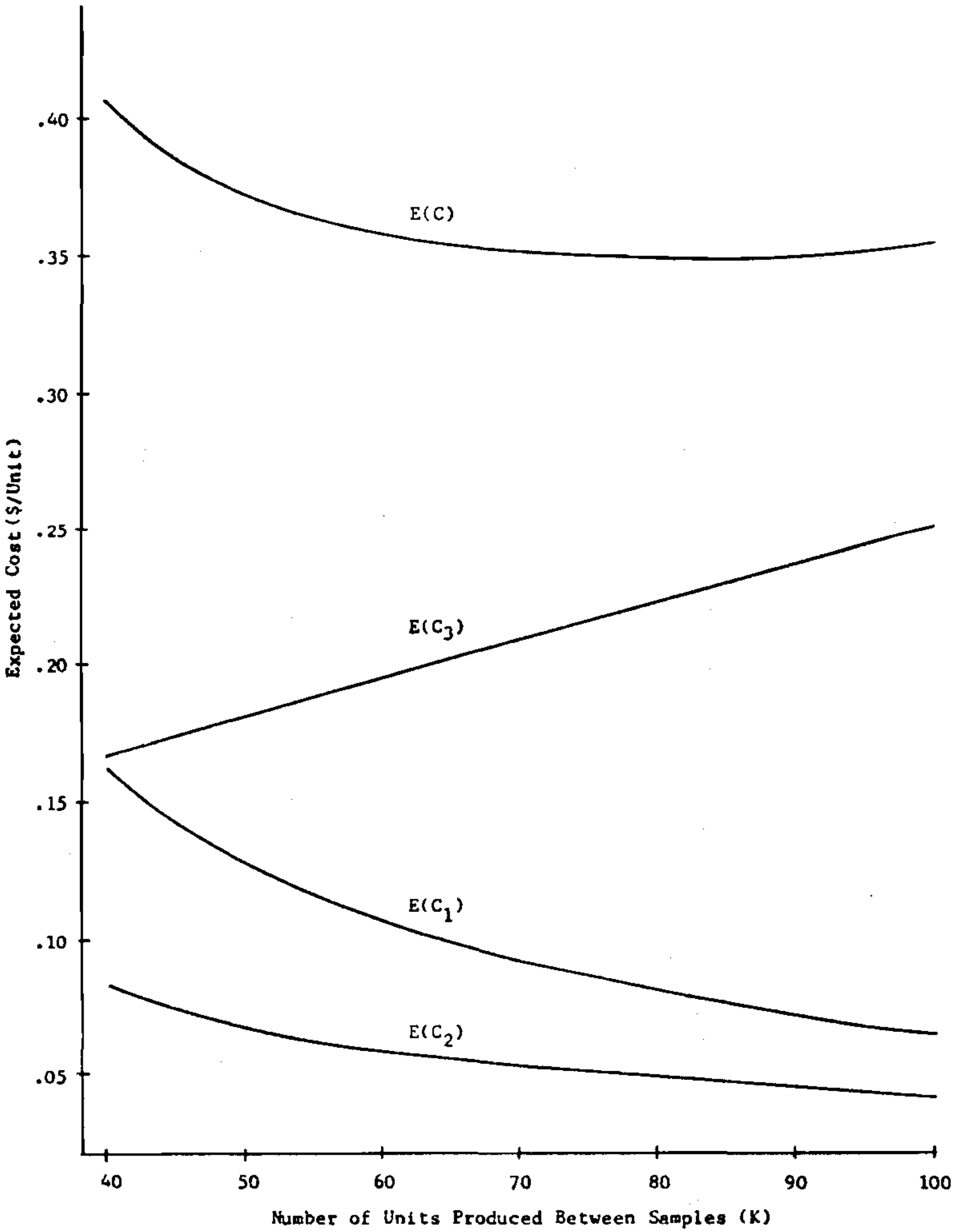


Figure 6. Expected Costs Versus Number of Units Produced Between Samples with $N=14$, $L=1.5$, $\lambda'=1000$, $S=6$, $\pi=.376$, $\underline{p}=\underline{p}_2$, $a_1=5.0$, $a_2=0.1$, $a_3=20.0$, and $a_4=10.0$

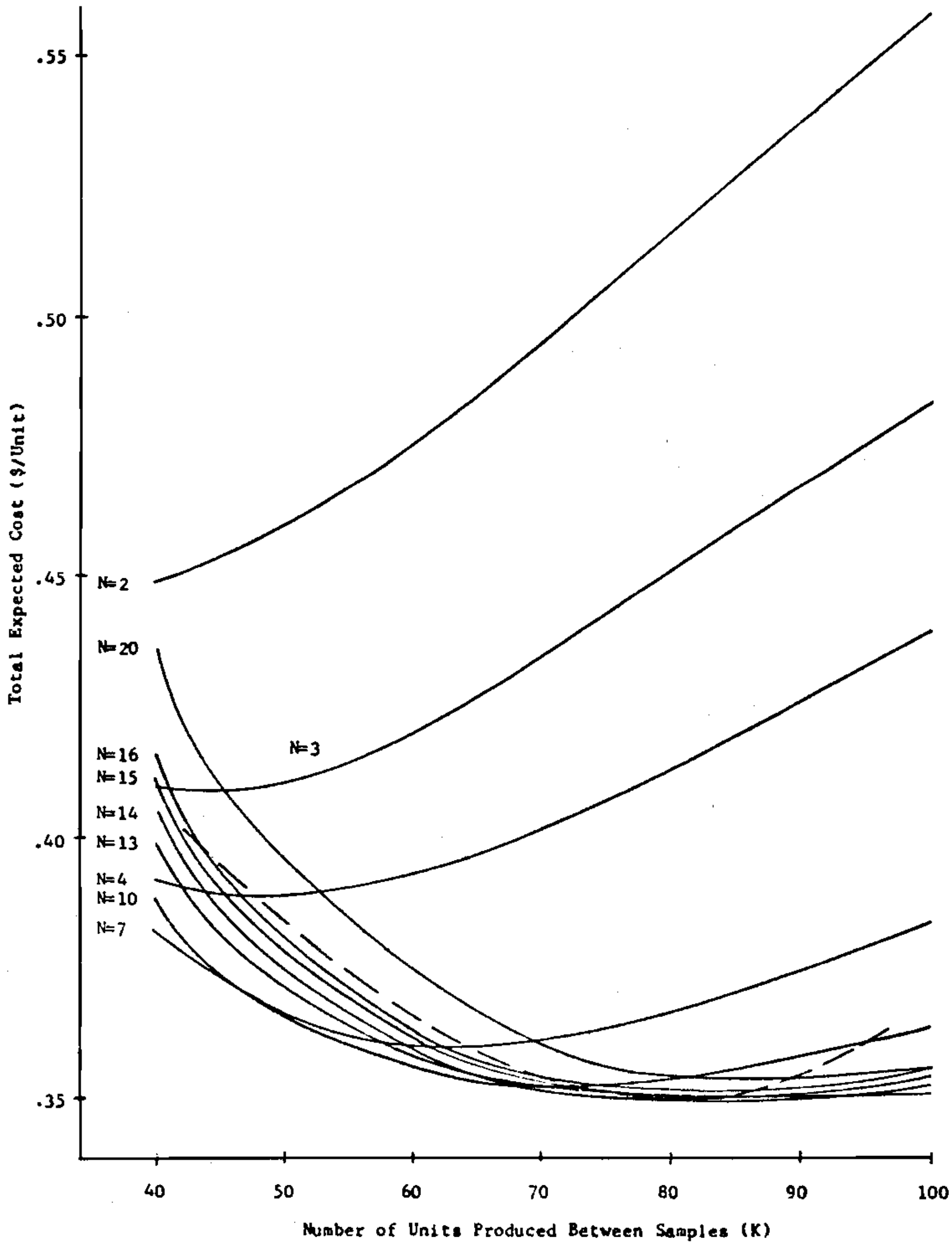


Figure 7. Total Expected Cost Versus N and K with L=1.5, λ'=1000, S=6, π=.376, $\underline{p}$=$\underline{p}_2$, a_1=5.0, a_2=0.1, a_3=20.0, and a_4=10.0

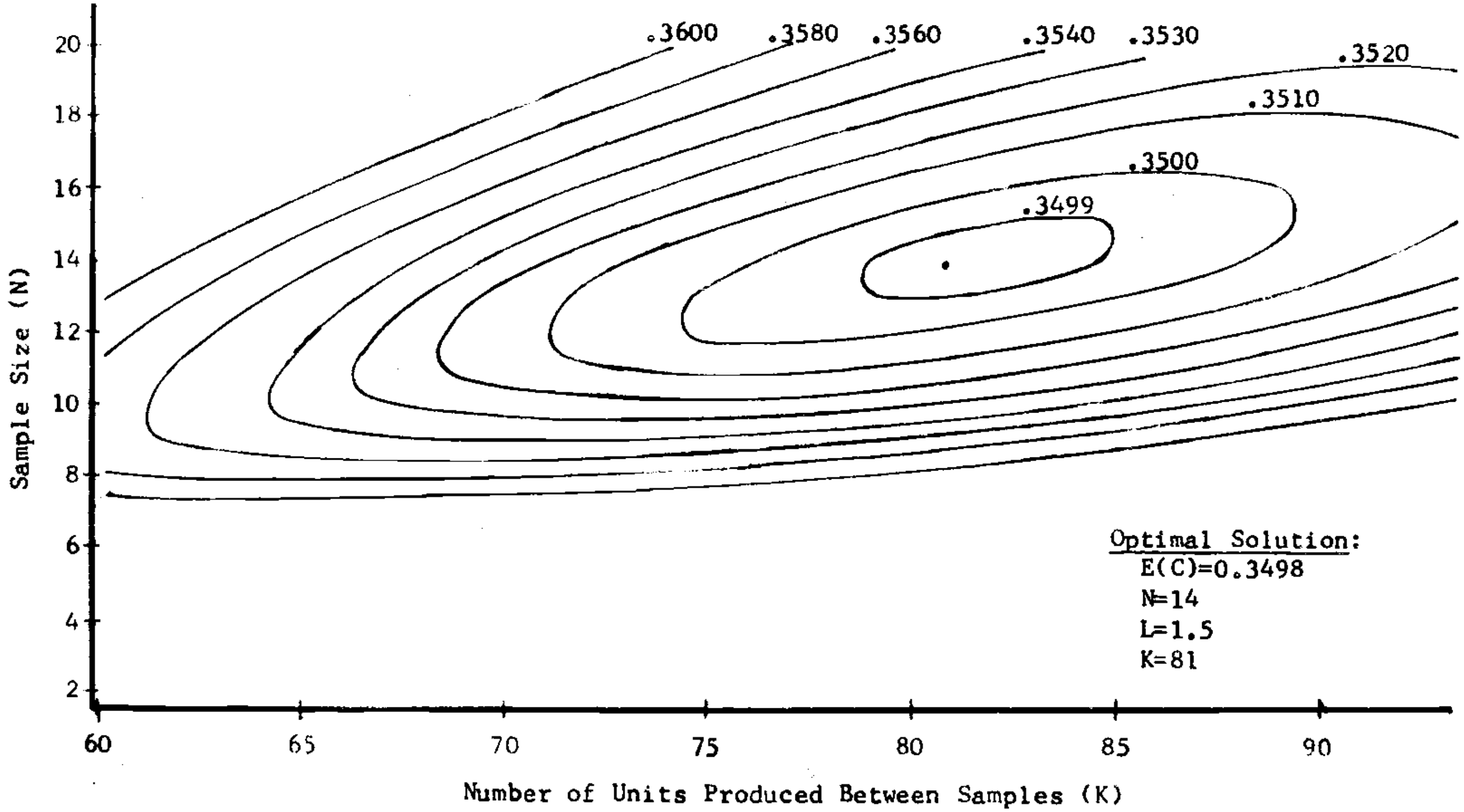


Figure 8. Total Expected Cost (\$/Unit) Versus N and K with L=1.5, $\lambda'=1000$, S=6, $\pi=.376$, $\underline{p}=\underline{p}_2$, $a_1=5.0$, $a_2=0.1$, $a_3=20.0$, and $a_4=10.0$

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

4.1 Conclusions

Based on several limiting assumptions, the model presented in Chapter II can be utilized to develop a minimum cost fraction defective control chart for a wide variety of production processes. The model can evaluate a change in quality of the process output subject to random shifts in the fraction defective by a simple attribute inspection plan. The optimal sampling plan in terms of E(C), N, L, and K can be used to establish an efficient quality control procedure. It is recognized that the simplicity of the model may limit the accuracy of the optimal solution, but it is also recognized that this inherent simplicity will greatly aid the acceptance and use of the model. The results from this investigation indicate:

(1) The optimal values of N and K are sensitive to changes in the fixed cost per sample or in the mean shift of a process.

(2) With $\underline{p} = \underline{p}_2$, the p chart model is relatively insensitive to large changes in L. However, the expected cost of the model changes when the value of L defines different criteria for rejection in the control procedure.

(3) While the convexity of the expected cost surface cannot be proven for all cases, the results tend to support the assumption of local convexity of the cost surface as both N and K vary over limited ranges.

(4) The total expected cost associated with the quality control

procedure is relatively insensitive to the number of out of control states.

(5) The optimal value of K is dependent upon the mean deterioration rate, while the optimal sample size and control limits are not.

A brief comparison between the optimal solutions obtained using the Knappenberger and Grandage cost model based on either a measurement sampling plan ($\overline{X}$ chart) or an attribute sampling plan (p chart) was presented. The degree to which the $\overline{X}$ chart model has an economic advantage over the p chart model depends on the relative values of a_1 and π. This economic advantage of the $\overline{X}$ chart model tends to decrease when either a_1 or π increases or both.

The sensitivity analysis demonstrated the importance of accurate values for the model parameters and cost coefficients on the optimal solution. The prior information required for this model, while easily obtained, greatly influenced the optimal solution. If the accuracy of this prior information for the process is doubtful or if the production process is modified, then the model should be modified to account for different values of the parameters and cost coefficients, and the optimal solutions will reflect the changes in the test parameters.

While further improvements in the optimization technique can be made, the modified Hooke and Jeeves pattern search was shown to be more efficient than a grid search. To ensure the global optimum is located, the pattern search can be started from different initial values of the argument $\underline{X}$. When this procedure is required, the total computer time necessary to locate the global optimum should remain less than if a grid search is utilized.

4.2 Recommendations

With the completion of this investigation, several related topics requiring additional research are proposed.

(1) The behavior of the cost surface should be investigated further to determine its convexity and to locate the optimal region of interest from direct consideration of the model parameters.

(2) Some initial work was completed in this investigation on the effect of varying the number of out of control states (S). The method used to determine the fractions defective vector for a process with S = 1 or 2 from the same process with S = 6 is only one of many possible methods. Other methods to compute $\underline{p}$ with only one or two out of control states remain to be investigated. If a good method is found, then a process with S = 6 can be approximated with only one or two out of control states over a wide range of cost coefficients and model parameters.

(3) The expected cost of investigating and correcting the process, a_3, was assumed to be independent of the true parameter p. However, it may be worthwhile to investigate the effect of assuming the value of a_3 is directly proportioned to the expected value of a shift in the fraction defective given a shift occurs.

(4) The relationship between the number of defects produced and the cost to the manufacturer for each defective produced, a_4, was assumed to be linear. The exact relationship in this case is difficult to determine, but it seems reasonable to believe that as the percentage of defectives produced increases the cost per defective will also increase. Therefore, a method which expresses a_4 as a function of the fraction defective would be an alternate assumption worthy of further study.

APPENDICES

APPENDIX A

CONVERSION OF MODEL TO FORTRAN V

Computer Symbology

Model Symbol	Computer Symbol
a_1	SA(1)
a_2	SA(2)
a_3	SA(3)
a_4	SA(4)
λ	PLAMDA
R	RATE
P_0	PO
S	IS
π	PI
λ'	CLAMDA
N	SN=X(1)
L	SL=X(2)
K	SK=X(3)
$\underline{p}$	P(7)
$\underline{q}$	Q(7)
$\underline{c}$	C(7)
$\underline{\gamma}$	G(7)
$\underline{\alpha}$	A(7)
$\underline{f}$	F(7)
$E(C_1)$	EC(1)·SA(4)

Model Symbol	Computer Symbol
$E(C_2)$	EC(2)·SA(4)
$E(C_3)$	EC(3)·SA(4)
E(C)	Y
F	FR

```
      DIMENSION X(3),DEL(3),BASEO(3),BASEN(3)
      COMMON BMIN(3),BMAX(3),DELTA(3),NE,M,
     X  P(7),Q(7),C(7),A(7),G(7),F(7),
     X  B(7,7),BB(7,7),SA(4),PLAMDA,RATE,IS,PI,NLCL,NUCL,
     X  RLCL,RUCL
      NE=0
C STARTING POINT,INITIAL STEP SIZE, TERMINATING STEP SIZE
C UPPER AND LOWER LIMITS
      WRITE(6,105)
      DO 5 I=1,3
      READ(5,600) BASEN(I), DELTA(I), DEL(I),BMAX(I),BMIN(I)
      WRITE(6,106) I,BASEN(I),DELTA(I),DEL(I),BMAX(I),BMIN(I)
    5 CONTINUE
      READ(5,601) NUMBER
      READ(5,602)(SA(I),I=1,4)
      READ(5,603) PLAMDA, RATE
      READ(5,604)  IS, PI
      READ(5,605)(P(I),I=1,7)
  602 FORMAT(4F10.0)
  603 FORMAT(2F10.0)
  604 FORMAT(I10,F10.0)
  601 FORMAT(I10)
  605 FORMAT(7F10.0)
  118 FORMAT(/4X,8HMAX. NE=,I4)
      WRITE(6,118) NUMBER
      WRITE(6,300)(SA(I),I=1,4)
      WRITE(6,301) PLAMDA, RATE
      WRITE(6,302)  IS, PI
  300 FORMAT(2X,6HSA(1)=,F6.2,2X,6HSA(2)=,F6.2,2X,6HSA(3)=,
     XF6.1,2X,6HSA(4)=,F6.1)
  301 FORMAT(2X,7HPLAMDA=,F5.3,2X,5HRATE=,F5.1)
  302 FORMAT(2X,3HIS=,I2,2X,3HPI=,F5.3)
  600 FORMAT(5F10.0)
      WRITE(6,107)
      WRITE(6,108)
      WRITE(6,109)
      WRITE(6,110)
    1 DO 10 I=1,3
      X(I)= BASEN(I)
   10 CONTINUE
      WRITE(6,111)
      CALL MAIN(X,FXBN)
C STARTING FUNCTION VALUE
      FX=FXBN
      IF(NE .GE. NUMBER) GO TO 100
C EXPLORATORY MOVES
      WRITE(6,113)(DELTA(I),I=1,3)
      CALL EXPLMV(X,FX)
      WRITE(6,114) FX,X
      IF(FX .GE. FXBN ) GO TO 3
C  SET NEW BASE POINT
    2 DO 20 I=1,3
      BASEO(I)=BASEN(I)
      BASEN(I)=X(I)
   20 CONTINUE
      FXBN=FX
      WRITE(6,111)
      WRITE(6,112) FXBN,X
      WRITE(6,115)
C PATTERN MOVE
      DO 21 I=1,3
      X(I)=BASEN(I)*2.0 - BASEO(I)
      IF(X(I) .GT. BMAX(I))  X(I)=BMAX(I)
      IF(X(I) .LT. BMIN(I))  X(I)=BMIN(I)
   21 CONTINUE
      CALL MAIN(X,FX)
      IF(NE .GE. NUMBER) GO TO 100
C EXPLORATORY MOVES
      WRITE(6,113)(DELTA(I),I=1,3)
      CALL EXPLMV(X,FX)
```

```
      WRITE(6,114) FX,X
      IF(FX .LT. FXBN) GO TO 2
C PATTERN MOVE HAS FAILED
      GO TO 1
    3 CONTINUE
C CHECK CURRENT STEP SIZE
C IF IT IS SMALL ENOUGH STOP
C IF IT IS LARGE, REDUCE IT ONE-HALF AND GO BACK
      IF(DELTA(1) .GE. 2.0*DEL(1)) GO TO 31
      DELTA(1)=DEL(1)
   40 IF(DELTA(2) .GE. 2.0*DEL(2)) GO TO 32
      DELTA(2) = DEL(2)
   41 IF(DELTA(3) .GE. 2.0*DEL(3)) GO TO 33
      DELTA(3)=DEL(3)
      GO TO 100
   31 DELTA(1)=DELTA(1)*0.5
      GO TO 40
   32 DELTA(2)=DELTA(2)*0.5
      GO TO 41
   33 DELTA(3)=DELTA(3)*0.5
      GO TO 1
  100 WRITE(6,116)
      K=IS+1
      CALL MAIN(BASEN,FXBN)
      WRITE(6,117) NE,FXBN,BASEN
      WRITE(6,310) RLCL, RUCL
  310 FORMAT(/10X,5HRLCL=,F7.3,4X,5HRUCL=,F7.3)
      XSL=(NUCL-X(1)*P(1))/(SQRT(P(1)*(1.0-P(1))*X(1)))
      SSL=1/(SQRT(P(1)*(1.0-P(1))*X(1)))-XSL
      WRITE(6,202) NLCL, NUCL
      WRITE(6,311) XSL, SSL
  311 FORMAT(/10X,7HMAX SL=,F7.3,2X,7HMIN SL=,F7.3)
  202 FORMAT(/10X,5HNLCL=,I3,8X,5HNUCL=,I3)
      WRITE(6,204)
      DO 400 I=1,K
      WRITE(6,203)(BB(I,J), J=1,K)
  400 CONTINUE
      WRITE(6,205)
      DO 401 I=1,K
      WRITE(6,203)(B(I,J), J=1,K)
  401 CONTINUE
  203 FORMAT(/2X,7F8.4)
  204 FORMAT(/8X,8HB MATRIX)
  205 FORMAT(/8X,16HB INVERSE MATRIX)
      WRITE(6,201)
  201 FORMAT(///'  PRINT THE VECTORS  P,Q,C,A,G,F ')
  200 FORMAT(/4X,7F9.4)
      WRITE(6,200) (P(I),I=1,7)
      WRITE(6,200) (Q(I),I=1,7)
      WRITE(6,200) (C(I),I=1,7)
      WRITE(6,200) (A(I),I=1,7)
      WRITE(6,200) (G(I),I=1,7)
      WRITE(6,200) (F(I),I=1,7)
  105 FORMAT('    DATA:        INIT PT  DELTA    DEL      MAX       MIN')
  106 FORMAT(/4X,I1,5X,5F9.3)
  107 FORMAT(/2X,'NO. OF           EXPECTED          CONTROL UNITS')
  108 FORMAT(2X,'EVAL.            COST PER   SAMPLE CHART   BETWEEN')
  109 FORMAT(2X,'OF MAIN          UNIT        SIZE   LIMITS  SAMPLES')
  110 FORMAT('      NE              Y           SN      SL        SK         VAR
     1')
  111 FORMAT('  BELOW IS THE COST AT THE NEW BASE POINT (FXBN)')
  112 FORMAT(/16X,F10.4,F8.1,F8.3,F8.1)
  113 FORMAT(/2X,22HEXPLORE MOVE DELTA(1)=,F5.3,10H DELTA(2)=,
     1  F5.3,10H DELTA(3)=,F5.3)
  114 FORMAT(/13X,3HFX=,F10.4,F8.1,F8.3,F8.1)
  115 FORMAT('  PATTERN MOVE')
  116 FORMAT(//'   OPTIMAL SOLUTION  COST       SN       SL       SK')
  117 FORMAT(/4X,I3,4X,5HFXBN=,F10.4,F8.1,F8.3,F8.1)
      END
```

```
      SUBROUTINE EXPLMV(X,FX)
      DIMENSION X(3)
      COMMON BMIN(3),BMAX(3),DELTA(3),NE,M,
     X  P(7),Q(7),C(7),A(7),G(7),F(7),
     X  B(7,7),BB(7,7),SA(4),PLAMDA,RATE,IS,PI,NLCL,NUCL,
     X  RLCL,RUCL
      M=1
      X2=X(2)
      QN=Q(1)
      DO 201 I=1,3
   25 X(I)=X(I)+DELTA(I)
      RU=X(I)
      IF(X(I) .GT. BMAX(I))  X(I)=BMAX(I)
      CALL MAIN(X,FXI)
      IF(FXI .LT. FX ) GO TO 200
      IF(RU .GE. BMAX(I)) GO TO 20
      IF(I .EQ. 2 .AND. QN .EQ. Q(1))  GO TO 25
      X(I)=X(I)-2.0*DELTA(I)
      IF(I .EQ. 2)  X(2)=X2-DELTA(2)
      GO TO 10
   20 X(I)=RU - 2.0*DELTA(I)
      IF(I .EQ. 2)  X(2)=X2-DELTA(2)
   10 RL = X(I)
      GO TO 27
   26 X(2)=X(2)-DELTA(2)
      RL=X(2)
   27 IF(X(I) .LT. BMIN(I)) X(I)=BMIN(I)
      CALL MAIN(X,FXI)
      IF(FXI .LT. FX) GO TO 200
      IF(RL .LE. BMIN(I)) GO TO 21
      IF(I .EQ. 2 .AND. QN .EQ. Q(1))  GO TO 26
      X(I)=X(I)+DELTA(I)
      IF(I .EQ. 2)  X(2)=X2
      GO TO 202
   21 X(I)=RL + DELTA(I)
      IF(I .EQ. 2)  X(2)=X2
      GO TO 202
  200 FX=FXI
      QN=Q(1)
  202 CONTINUE
      M=M+1
  201 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE MAIN(X,Y)
      DIMENSION EC(4),
     1    CA(3), CC(6), FAC(6), SC(7), SUMA(7),
     2  V(1) , JC(7), DIAG(7),X(3)
      COMMON BMIN(3),BMAX(3),DELTA(3),NE,M,P(7),Q(7),C(7),
     X  A(7),G(7),F(7),
     X  B(7,7),BB(7,7),SA(4),PLAMDA,RATE,IS,PI,NLCL,NUCL,RLCL,RUCL
      K = IS + 1
      DO 1 I=1,K
      JC(I)=I
    1 CONTINUE
      V(1)=1.
C CALCULATE  CLAMDA
      CLAMDA = PLAMDA / RATE
C SET INITIAL VALUES OF SN, SL, SK
      SN = X(1)
      SL = X(2)
      SK = X(3)
C CALCULATE  CK
      CK=CLAMDA* SK
C CALCULATE CA(I)
      DO 5 I= 1,3
      CA(I) = SA(I) * (CLAMDA / SA(4))
    5 CONTINUE
C CALCULATE EC(1)
       EC(1) = CA(1) /CK + ( CA(2) * SN ) / CK
C CALCULATE CONTROL LIMITS
      N = SN
      D = SQRT ( P(1) * ( 1.0 - P(1) ) * SN)
      RLCL = SN * P(1) - SL * D
      NLCL=RLCL
      IF(RLCL .GE. 0.0)  GO TO 6
      NLCL=0
      DO 7 I=1,K
      Q(I)=0.0
    7 CONTINUE
      GO TO 8
    6 CONTINUE
      DO 9 I=1,K
      Q(I)=BIN(NLCL,N,P(I))
    9 CONTINUE
    8 CONTINUE
      RUCL=SN*P(1)+SL*D
      NUCL=RUCL
      RRUCL=NUCL
      IF(RRUCL .LT. RUCL)  GO TO 11
      NU=NUCL-1
      GO TO 12
   11 CONTINUE
      NU=NUCL
      NUCL=NUCL+1
   12 CONTINUE
C  CALCULATE  Q(I) I=1,K
      DO 13 I=1,K
      QU=1.0 -BIN(NU,N,P(I))
      Q(I)=Q(I)+QU
   13 CONTINUE
C CALCULATE C(I)  I = 1, K
      C(1) = 1.0 / ( EXP ( CLAMDA * SK ))
      Z1 = 1.0 - C(1)
      Z2 = ( 1.0 - ( 1.0 - PI ) ** IS)
C CALCULATE FACTORIALS
      FAC(1) = 1.0
      DO 14 I = 2, IS
      AI = I
      FAC(I) = FAC( I- 1) * AI
   14 CONTINUE
      CS= Z1 * FAC (IS) / Z2
      DO 15 I =2, IS
      CC(I) = ( PI **(I- 1 )) * (( 1.0 - PI ) ** ( K - I))
```

```
      C(I) = CS* CC(I) /(FAC(I-1 ) * FAC( K - I))
   15 CONTINUE
      CC(K)=(PI ** IS) * 1.0
      C(K)=CS *CC(K) / (FAC(IS)*1.0)
C CALCULATE B MATRIX (S+1)(S+1)
      DO 19 I=1,7
      DO 19 J=1,7
      B(I,J)=0.0
   19 CONTINUE
C  B(1,J)  J= 1,K
      DO 20 I = 1 , K
      B(1,I) = C(I)
   20 CONTINUE
C CALCULATE Q(I) * C(J) I = 2, K  :  J = 1, K
      DO 30 I = 2, K
      DO 30 J = 1, K
      B(I,J) = Q(I) * C(J)
   30 CONTINUE
C CAL B(I,J) I<J: J = 3, K  :  I = 2, J-1
      DO 40 J = 3, K
      L = J - 1
      DO 40 I = 2 , L
      B(I,J) = B(I,J) + (( 1.0 - Q(I)) * C(J)) / ZI
   40 CONTINUE
C CAL B(I,I) I=2, K
C    SUM C(I) I=1, K
      SC(1) =0.0
      DO 50 I = 2, K
      SC(I) = SC(I-1) + C(I)
      B(I,I) = B(I,I) + ( 1.0 - Q(I)) * SC(I) / ZI
   50 CONTINUE
      DO 51 I=1,K
      DO 51 J=1,K
      BB(I,J)= B(I,J)
   51 CONTINUE
C CALCULATE A(I)  I= 1, K
      DO 52 I= 1, K
C SUB. IDENTITY MATRIX
      B(I,I) = B(I,I) - 1.0
C ADD COLUMN OF 1 S AT COL S+1
      B(I,K) = 1.0
   52 CONTINUE
C FIND INVERSE OF B(I,J)
C CONDITION B(I,J): DIAGONAL ELEMENTS = 1.0
      K=IS+1
      DO 54 I = 1,K
      DIAG(I)=B(I,I)
      DO 53 J = 1,K
      B(I,J) = B(I,J) / DIAG(I)
   53 CONTINUE
   54 CONTINUE
C CALL MATHSTAT GJR
      CALL GJR(B , K ,K, K, K, $500 ,JC , V )
C CALCULATE VALUES OF A(I)  I= 1, K
      DO 57  J= 1 , K
      A(J)= B(K,J) / DIAG(J)
   57 CONTINUE
C CALCULATE EC(2)
      SPQA = 0.0
      DO 58 I = 1 , K
      SPQA=SPQA + Q(I) * A(I)
   58 CONTINUE
      EC(2) = SPQA * CA(3) / CK
C CALCULATE G(I) I = 1, K
C  FRACTION F
      FR = (1.0-(1.0 + CLAMDA * SK)* C(1)) / ( CLAMDA * SK * ZI)
C   SUM A(I)  J = 1, K
      SUMA(1) = 0.0
      DO 60 J = 2, K
      SUMA(J) = SUMA( J - 1 ) +  A(J)
```

```
   60 CONTINUE
      G(1) = A(1) * C(1) + FR * A(1) * Z1
      G(2) = A(2) * C(2)/Z1 + A(1) * ( 1.0 - FR) * C(2) + A(2) * FR
     1   *( SC ( K ) - SC (2)) / Z1
      G(K)  = A(K)   * SC(K)   / Z1 + A(1) *( 1.0 - FR ) * C(K)  +
     1 C(K)   * ( 1.0 - FR ) * SUMA(K  - 1) / Z1
      DO 65 I = 3, IS
      G(I) = A(I) * SC(I) / Z1 + A(1) * ( 1.0 - FR ) * C(I) +
     1  C(I) * ( 1.0 - FR) * SUMA(I- 1) / Z1 + A(I) * FR *
     2  ( SC ( K  ) - SC(I)) / Z1
   65 CONTINUE
C CALCULATE F(I) I = 1, K
      DO 72 I=1,K
      F(I)=P(I)
   72 CONTINUE
C CALCULATE EC(3)
      SPFG = 0.0
      DO 80 I = 1 , K
      SPFG = SPFG + F(I) * G(I)
   80 CONTINUE
      EC(3) = SPFG
C CALCULATE EC(4)
      EC(4) = ( EC(1) + EC(2) + EC(3))
      Y=EC(4)*SA(4)
      NE=NE+1
      WRITE(6,205) NE,Y,X,M
      WRITE(6,207) (EC(I),I=1,4)
  205 FORMAT(/4X,I3,9X,F10.4,F8.1,F8.3,F8.1,3X,I4)
  207 FORMAT(10X,6HEC(1)=,F6.4,2X,6HEC(2)=,F6.4,2X,6HEC(3)=,
     X  F6.4,2X,6HEC(4)=,F7.4)
      GO TO 550
  500 WRITE ( 6, 501)
  501 FORMAT ( ' ERROR IN GJR PROGRAM ' )
  550 CONTINUE
      RETURN
      END
```

APPENDIX B

CRITERIA FOR REJECTION

The criteria for rejection, $D \le [N(LCL)]^-$ or $D \ge [N(UCL)]^+$, as defined in section 2.3.1, represent the values of D when the null hypothesis is rejected. L_{max} represents the value of L which exactly yields the indicated integer value of $[N(UCL)]^+$. If the value of L is decreased from L_{max} to L_{min}, the size of the critical region as defined by $[N(UCL)]^+$ and $[N(LCL)]^-$ will not change. Therefore, the criteria for rejection remain the same. When L is varied outside the range, $L_{min} \le L \le L_{max}$, the criteria for rejection and the size of the critical region change according to the rules in section 2.3.1. The values of L_{max} and L_{min} for the possible criteria for rejection in the control procedure with $\underline{p}_2$ and $2 \le N \le 30$ are listed in Table 12.

Table 12. Values of L_{max} and L_{min} and of the Criteria for Rejection in the Control Procedure with $2 \leq N \leq 30$ and $\underline{p}=\underline{p}_2$

N	N(UCL)= 1.0	N(UCL)= 1.0 / N(UCL)= 2.0	N(UCL)= 2.0
	L_{min}	L_{max} L_{min}	L_{max}
2	.1407	6.965	
3	.1750	5.629	
4	.2011	4.824	
5	.2244	4.270	
6	.2464	3.857	
7	.2664	3.533	
8	.2843	3.269	
9	.3015	3.049	6.398
10	.3178	2.860	6.039
11	.3333	2.697	5.727
12	.3481	2.553	5.454
13	.3624	2.425	5.212
14	.3760	2.310	4.996
15	.3892	2.206	4.800
16	.4020	2.111	4.623
17	.4144	2.023	4.461
18	.4264	1.943	4.312
19	.4381	1.868	4.173
20	.4494	1.798	4.045
21	.4605	1.732	3.925
22	.4714	1.671	3.814
23	.4821	1.614	3.710
24	.4930	1.559	3.611
25	.5020	1.508	3.518
26	.5120	1.459	3.430
27	.5220	1.412	3.346
28	.5310	1.368	3.267
29	.5410	1.325	3.191
30	.5510	1.284	3.119

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